

A FRÉCHET TOPOLOGY ON MEASURED LAMINATIONS AND EARTHQUAKES IN THE HYPERBOLIC PLANE

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ABSTRACT. We prove that the bijective correspondence between the space of bounded measured laminations $ML_b(\mathbb{H})$ and the universal Teichmüller space $T(\mathbb{H})$ given by $\lambda \mapsto E^\lambda|_{S^1}$ is a homeomorphism for the Fréchet topology on $ML_b(\mathbb{H})$ and the Teichmüller topology on $T(\mathbb{H})$, where E^λ is an earthquake with earthquake measure λ . A corollary is that earthquakes with discrete earthquake measures are dense in $T(\mathbb{H})$. We also establish infinitesimal versions of the above results.

1. INTRODUCTION

A Riemann surface is said to be *hyperbolic* if its universal covering is the hyperbolic plane \mathbb{H} .¹ A quasiconformal map between two hyperbolic Riemann surfaces lifts to a quasiconformal map between their universal coverings, which are identified with the hyperbolic plane \mathbb{H} . This map continuously extends to a quasisymmetric map of the boundary $\partial\mathbb{H}$ of the hyperbolic plane, which is in turn identified with the unit circle S^1 . The homotopy class of a quasiconformal map between two Riemann surfaces is uniquely determined by the quasisymmetric map of S^1 , and this induces a natural complex analytic embedding of the Teichmüller space of any hyperbolic Riemann surface into the Teichmüller space $T(\mathbb{H})$ of the hyperbolic plane \mathbb{H} , called the *universal Teichmüller space*.

The universal Teichmüller space $T(\mathbb{H})$ is the space of all quasisymmetric maps of the unit circle S^1 modulo post-composition by Möbius maps which preserve \mathbb{H} . It is an infinite-dimensional complex Banach manifold which contains other interesting spaces of circle maps. We study $T(\mathbb{H})$ by the use of the hyperbolic geometry of \mathbb{H} . Our main objects are earthquakes in the hyperbolic plane \mathbb{H} and Hölder distributions on the space \mathcal{G} of geodesics of the hyperbolic plane \mathbb{H} .

Earthquake maps in the hyperbolic plane \mathbb{H} (and on any hyperbolic Riemann surface) were introduced by Thurston [20]. An earthquake in the hyperbolic plane is a bijective map $E : \mathbb{H} \rightarrow \mathbb{H}$ which is *supported* on a geodesic lamination \mathcal{L} in \mathbb{H} in the sense that it is a hyperbolic isometry on each *stratum* (i.e. a leaf of \mathcal{L} or a component of $\mathbb{H} \setminus \mathcal{L}$) of \mathcal{L} , and which (relatively) translates to the left points of different strata of \mathcal{L} . An earthquake $E : \mathbb{H} \rightarrow \mathbb{H}$ continuously extends to a homeomorphism of S^1 and it induces a transverse Borel measure to its support lamination \mathcal{L} , called the *earthquake measure*. The earthquake measure of E measures the amount of the

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¹We are particularly interested in the geometrically infinite hyperbolic Riemann surfaces, e.g. the hyperbolic plane \mathbb{H} , an infinite genus surface, a surface with an interval of ideal boundary points. All these surfaces have infinite hyperbolic area.

relative movement to the left by E . An earthquake measure λ uniquely determines earthquake $E^\lambda : \mathbb{H} \rightarrow \mathbb{H}$ up to post-composition by Möbius maps.

Thurston [20] showed that any homeomorphism of the unit circle S^1 is obtained as the continuous extension of an earthquake in \mathbb{H} to its boundary S^1 . In other words, any homeomorphism of S^1 can be geometrically constructed as the continuous extension to the boundary S^1 of a piecewise isometry of \mathbb{H} which moves strata of its support geodesic lamination to the left by the amount given by a transverse Borel measure to the lamination. However, the relationship between homeomorphisms and earthquake measures of the earthquakes inducing them is not a simple one. This paper is mainly concerned with the dependence of the earthquake measures on homeomorphisms of S^1 .

A measured lamination λ is said to be *bounded* if

$$\sup_I \lambda(I) < \infty$$

where the supremum is over all geodesic arcs I of unit length that transversely intersect the support of λ . Then a homeomorphism is quasimetric if and only if $h = E^\lambda|_{S^1}$ for a bounded earthquake measure λ (see [7], [12] and [14]).

We denote by $ML_b(\mathbb{H})$ the space of all bounded measured laminations. The above statement gives a well-defined *earthquake measure map*

$$\mathcal{EM} : T(\mathbb{H}) \rightarrow ML_b(\mathbb{H})$$

by $\mathcal{EM}([h]) = \lambda$, where quasimetric map h is obtained by continuously extending to S^1 earthquake E^λ with earthquake measure λ . The earthquake measure map is a bijection by the above. Our main result establishes a natural topology on $ML_b(\mathbb{H})$ for which \mathcal{EM} is a homeomorphism.

Each oriented geodesic in \mathbb{H} is uniquely determined by the pair of its endpoints on S^1 , the initial point and the terminal point. Then the space \mathcal{G} of unoriented geodesics in \mathbb{H} is isomorphic to $(S^1 \times S^1 \setminus \text{diag}) / \sim$, where $(a, b) \sim (b, a)$ and $\text{diag} = \{(a, a) | a \in S^1\}$. We fix an angle metric on S^1 and obtain an induced metric d on \mathcal{G} . Let $H\ddot{o}l_0$ be the space of all Hölder continuous functions $\varphi : \mathcal{G} \rightarrow \mathbb{R}$ with compact support. For $0 < \nu \leq 1$, let $H\ddot{o}l_0^\nu$ be the space of all ν -Hölder continuous functions $\varphi : \mathcal{G} \rightarrow \mathbb{R}$ with compact support. Let $Q^* = ([-i, 1] \times [i, -1]) / \sim \subset \mathcal{G}$.

Let $\text{test}(\nu)$ be the space of pairs (φ, Q) with the following properties. The function $\varphi : \mathcal{G} \rightarrow \mathbb{R}$ is ν -Hölder continuous and its support is contained in $Q = ([a, b] \times [c, d]) / \sim$. The closed arcs $[a, b], [c, d] \subset S^1$ are disjoint and the *Liouville measure* $L(Q) := \log \frac{(a-c)(b-d)}{(a-d)(b-c)}$ of Q equals $\log 2$. If $\gamma_Q : Q^* \rightarrow Q$ is a Möbius map, then

$$\|\varphi \circ \gamma_Q\|_\nu < \infty$$

where $\|\varphi\|_\nu$ is the ν -Hölder norm of φ (cf. §2.4).

The space \mathcal{H} of Hölder distributions consists of all linear functionals $W : H\ddot{o}l_0 \rightarrow \mathbb{R}$ such that

$$\|W\|_\nu := \sup_{(\varphi, Q)} |W(\varphi)| < \infty$$

for each ν , $0 < \nu \leq 1$, where the supremum is over all $(\varphi, Q) \in \text{test}(\nu)$. The family of ν -norms on \mathcal{H} induces a Fréchet structure on \mathcal{H} . The space of Hölder distributions for closed surfaces is introduced by Bonahon [2], and generalized in the above form for geometrically infinite surfaces [13]. The Liouville map $\mathcal{L} : T(\mathbb{H}) \rightarrow \mathcal{H}$ given by the pull-backs of the Liouville measure is an analytic homeomorphism onto its

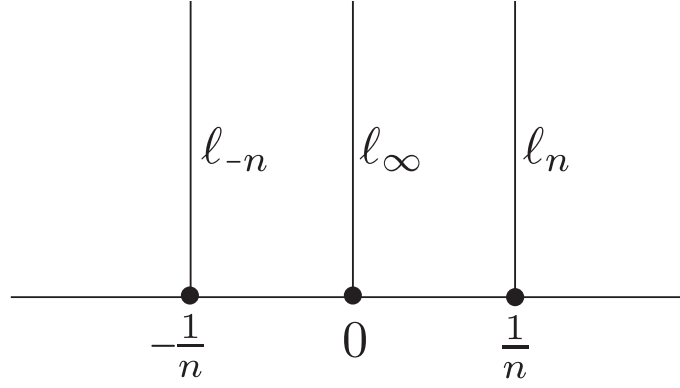
FIGURE 1. $\lambda_n \rightharpoonup \lambda$ in the Fréchet topology.

image (cf. [1], [16], [11]). Bonahon [1] defined Thurston boundary to Teichmüller spaces of closed surfaces using the Liouville map and his construction extends to geometrically infinite surfaces [16].

Our main result makes a connection between the Fréchet topology on \mathcal{H} and earthquake maps in the hyperbolic plane. Namely, we show that the induced Fréchet topology on $ML_b(\mathbb{H}) \subset \mathcal{H}$ is capturing the subtleties of the Teichmüller topology on $T(\mathbb{H})$ and the earthquake maps in the hyperbolic plane \mathbb{H} .

Theorem 1 (Earthquake measure map is a homeomorphism). *The earthquake measure map*

$$\mathcal{EM} : T(\mathbb{H}) \rightarrow ML_b(\mathbb{H})$$

is a homeomorphism for the Teichmüller topology of $T(\mathbb{H})$ and the Fréchet topology on $ML_b(\mathbb{H})$.

The above theorem also holds for any geometrically infinite Riemann surfaces by simply noting that a quasiconformal map which is invariant under a Fuchsian group is induced by an earthquake whose earthquake measure is invariant under the same Fuchsian group. In the case of a closed hyperbolic surface S , Kerckhoff [10] showed that the earthquake measure map is a homeomorphism for the weak* topology on $ML(S)$. Using the techniques in the paper, it is easy to prove that $\mathcal{EM} : M\ddot{o}b(\mathbb{H})/Homeo(S^1) \rightarrow ML(\mathbb{H})$ is a homeomorphism for the topology of pointwise convergence on the space of homeomorphisms $Homeo(S^1)$ of S^1 and the weak* topology on the (not necessarily bounded) measured laminations $ML(\mathbb{H})$ of \mathbb{H} , where $M\ddot{o}b(\mathbb{H})$ are Möbius maps that preserve \mathbb{H} . We note that the weak* topology on $ML_b(\mathbb{H})$ is strictly weaker than the Fréchet topology.

To illustrate the difference between the weak* topology and the Fréchet topology on $ML_b(\mathbb{H})$ we consider the following example. Identify the hyperbolic plane \mathbb{H} with the upper half-plane and its boundary $\partial\mathbb{H}$ with $\mathbb{R} = \mathbb{R} \cup \{\infty\}$. Let $l = (0, \infty)/\sim$ and $l_n = (\frac{1}{n}, \infty)/\sim$ be geodesics in \mathbb{H} . Let δ_l and δ_{l_n} denote the Dirac measures on \mathcal{G} with supports l and l_n , respectively. Then $\frac{\delta_{l_n} + \delta_{l-n}}{2}$ converges in the weak* topology to δ_l as $n \rightarrow \infty$, but it does *not* converge in the Fréchet topology (Figure ??). See §5 for further discussion and examples.

An earthquake is said to be *finite* if its earthquake measure has finite support in \mathcal{G} . Thurston [20] proved that the graph of any earthquake $E : \mathbb{H} \rightarrow \mathbb{H}$ is

approximated by the graphs of finite earthquakes. Gardiner-Hu-Lakic [7] proved that each monotone map from an n -tuple of points in S^1 into S^1 can be realized by a finite earthquake whose support geodesics are in the n -tuple (finite earthquake theorem). We say that an earthquake is *discrete* if the support of its earthquake measure is a discrete subset of \mathcal{G} . Next to finite earthquakes, discrete earthquakes are the simplest possible earthquakes and, by definition, finite earthquakes are discrete. We prove that each earthquake E can be approximated by a sequence of discrete earthquakes E_n in the sense that $E|_{S^1} \rightarrow E_n|_{S^1}$ in the Teichmüller topology as $n \rightarrow \infty$. Theorem below is a direct consequence of Theorem 5 (cf. §7.2) and Theorem 1.

Theorem 2 (Countable Earthquake Theorem). *Let ML_b^{disc} be the set of all bounded measured laminations whose supports are discrete subsets of \mathcal{G} . Then the set*

$$\{[E^\lambda|_{S^1}] : \lambda \in ML_b^{disc}\}$$

is a dense subset of $T(\mathbb{H})$ in the Teichmüller topology.

We prove analogous statements for the Zygmund vector fields and the infinitesimal earthquakes. Let V be a vector field on S^1 and let $Q = ([a, b] \times [c, d]) / \sim$, called a *box of geodesics*, be a subset of \mathcal{G} such that $[a, b] \cap [c, d] = \emptyset$. Define

$$V[Q] := \frac{V(a) - V(c)}{a - c} - \frac{V(a) - V(d)}{a - d} + \frac{V(b) - V(d)}{b - d} - \frac{V(c) - V(d)}{c - d}.$$

The *cross-ratio norm* $\|V\|_{cr}$ of a vector field V is defined by

$$\|V\|_{cr} := \sup_Q V[Q],$$

where the supremum is over all boxes of geodesics $Q = ([a, b] \times [c, d]) / \sim$ with $L(Q) = \log 2$. A vector field V on S^1 is *Zygmund bounded* if its cross-ratio norm $\|V\|_{cr}$ is finite. Let $\mathcal{Z}(S^1)$ be the vector space of all Zygmund bounded vector fields on S^1 modulo the closed subspace of quadratic polynomials. (Note that quadratic polynomials are infinitesimal deformations of the paths of Möbius maps.)

A vector field V on S^1 is Zygmund bounded if and only if there exists a differentiable path of quasimetric maps $t \mapsto h_t$, for $|t| < \epsilon$ with $\epsilon > 0$, such that $h_0 = id$ and $\frac{d}{dt}h_t|_{t=0} = V$ (see [6]). Given $\lambda \in ML_b(\mathbb{H})$, the path $t \mapsto E^{t\lambda}|_{S^1}$ is differentiable. Its derivative at $t = 0$ is a Zygmund bounded vector field, called the *infinitesimal earthquake*, and we denote it by

$$\dot{E}^\lambda|_{S^1} := \frac{d}{dt}(E^{t\lambda}|_{S^1})|_{t=0}.$$

Gardiner [5] proved that each Zygmund bounded vector field arises as an infinitesimal earthquake and he also established the formula (see also §9)

$$\dot{E}^\lambda|_{S^1} = \int_{\mathcal{G}} \dot{E}_\ell^\lambda d\lambda(\ell),$$

where $\dot{E}_\ell^\lambda(z) = \frac{(z-a)(z-b)}{a-b}$ for $z \in S^1$ with a and b the endpoints of ℓ such that the triple (a, z, b) has positive orientation on S^1 .

The *infinitesimal earthquake measure map*

$$\mathcal{EM} : ML_b(\mathbb{H}) \rightarrow \mathcal{Z}(S^1)$$

defined by

$$\mathcal{EM} : \lambda \mapsto \dot{E}^\lambda|_{S^1}$$

is a bijection. We prove that the Fréchet topology on $ML_b(\mathbb{H})$ makes \mathcal{EM} into a homeomorphisms analogous to the case of quasisymmetric maps.

Theorem 3 (Fréchet and Zygmund). *Let $\mathcal{ML}_b(\mathbb{D})$ be given the Fréchet topology and $\mathcal{Z}(S^1)$ be given the cross-ratio norm topology. Then, the infinitesimal earthquake measure map*

$$\mathcal{EM} : ML_b(\mathbb{H}) \rightarrow \mathcal{Z}(S^1)$$

is a homeomorphism.

An infinitesimal version of the countable earthquake theorem immediately follows from Theorem 5 in §7 and Theorem 3.

2. MEASURED LAMINATIONS AND HÖLDER DISTRIBUTIONS

2.1. Space of geodesics. Let \mathbb{D} be the unit disk model of the hyperbolic plane \mathbb{H} . The unit circle S^1 is identified with the set of ideal boundary points $\partial\mathbb{D}$ of the hyperbolic plane. Fix $z_0 \in \mathbb{D}$. Define the distance between $z_1, z_2 \in S^1$ to be smaller angle between the geodesic rays connecting z_0 with z_1 and z_2 , respectively. This gives an angle metric on S^1 which depends on z_0 . By varying $z_0 \in \mathbb{D}$ we obtain a biLipschitz class of metrics on S^1 .

A complete oriented geodesic g in \mathbb{D} is uniquely determined by an ordered pair of its distinct ideal endpoints on S^1 , the initial and the terminal point of g . Conversely, given an ordered pair of points on S^1 , there is a unique oriented hyperbolic geodesic with its initial endpoint being the first point and its terminal endpoint being the second point of the pair. Thus the space $\tilde{\mathcal{G}}$ of all oriented geodesics on \mathbb{D} is naturally identified with $S^1 \times S^1 \setminus \text{diag}$. Let \mathcal{G} be the set of all unoriented complete hyperbolic geodesic on \mathbb{D} . The set \mathcal{G} is identified with $(S^1 \times S^1 \setminus \text{diag}) / \sim$, where the equivalence is defined by $(a, b) \sim (b, a)$ and diag is the diagonal set of the product. We denote by $[a, b]$ the equivalence class of $(a, b) \in S^1 \times S^1 \setminus \text{diag}$. An angle metric d_{z_0} on S^1 with respect to $z_0 \in \mathbb{D}$ induces a metric \bar{d}_{z_0} on \mathcal{G} as follows. Let $[a, b], [c, d] \in \mathcal{G}$. Define $\bar{d}_{z_0}([a, b], [c, d]) = \min\{\max\{d_{z_0}(a, c), d_{z_0}(b, d)\}, \max\{d_{z_0}(a, d), d_{z_0}(b, c)\}\}$. The set of geodesics \mathcal{G} has a biLipschitz class of metrics obtained by varying $z_0 \in \mathbb{D}$.

A quasiconformal map $f : \mathbb{D} \rightarrow \mathbb{D}$ continuously extends to a quasisymmetric map $h : S^1 \rightarrow S^1$. Mori's theorem implies that h is a Hölder continuous homeomorphism of S^1 whose Hölder constant depends only on the maximal dilatation of f . Thus a quasisymmetric mapping of S^1 also induces a Hölder continuous homeomorphism of \mathcal{G} for the angle metric \bar{d}_{z_0} . Since each quasisymmetric map induces a biholomorphic isometry of the universal Teichmüller space, it is natural to work with the class of Hölder equivalent metrics to the metric \bar{d}_{z_0} . Recall that a metric d is Hölder equivalent to \bar{d}_{z_0} if there exist $C \geq 1$ and $0 < \nu \leq 1$ such that

$$\frac{1}{C}d([x, y], [x_1, y_1])^{\frac{1}{\nu}} \leq \bar{d}_{z_0}([x, y], [x_1, y_1]) \leq Cd([x, y], [x_1, y_1])^{\nu}.$$

2.2. Measured laminations. A *geodesic lamination* \mathcal{L} is a closed subset of \mathbb{D} together with a foliation by disjoint complete geodesics. We recall that the information of the foliation of the closed subset is necessary for the definition of a geodesic lamination in \mathbb{D} . For example, the hyperbolic plane can be foliated by complete hyperbolic geodesics in infinitely many different ways and each different foliation determines a different geodesic lamination. Equivalently, a geodesic lamination \mathcal{L} is a closed subset of \mathcal{G} such that no two geodesics in \mathcal{L} intersect in \mathbb{D} (they can have common ideal endpoints).

Each complete geodesic in \mathcal{L} is called a *leaf* of \mathcal{L} . A *stratum* of \mathcal{L} is either a geodesic of \mathcal{L} or a component of the complement of \mathcal{L} in \mathbb{D} .

A *measured lamination* λ is a positive, locally finite, Borel measure on the space of geodesics \mathcal{G} whose support $|\lambda|$ is a geodesic lamination. Each measured lamination λ induces a *transverse measure* to its support $|\lambda|$, namely an assignment of a positive, Borel measure to each closed finite hyperbolic arc I in \mathbb{D} whose support is $I \cap |\lambda|$ and which is invariant under homotopies which preserve the strata of $|\lambda|$. More precisely, the λ -mass of an arc I , denoted by $\lambda(I)$, is the λ -measure of the set of geodesics in \mathcal{G} which intersect I . Conversely, a transverse measure to a geodesic lamination \mathcal{L} determines a unique measured lamination λ whose support is $\mathcal{L} = |\lambda|$. For this correspondence we refer the reader to §1 of [2]. A measured lamination λ is *bounded* if the *Thurston's norm*

$$\|\lambda\|_{Th} = \sup_I \lambda(I)$$

is finite, where I runs over all geodesic arcs in \mathbb{D} with unit length. Let $\mathcal{ML}_b(\mathbb{D})$ be the set of bounded measured laminations on \mathbb{D} . When the support of a measured lamination λ consists of one geodesic, we say that λ is an *elementary measured lamination*.

Möbius transformations act isometrically on the set of bounded measured laminations by the *pull-backs* as follows. Let $\gamma \in \text{Möb}(\mathbb{D})$ and λ a measured lamination. We define $\gamma^*\lambda$ as the measured lamination with support $\gamma^{-1}(|\lambda|)$ and the transverse measure $\lambda \circ \gamma$, where $(\lambda \circ \gamma)(I) = \lambda(\gamma(I))$ for all geodesic arcs I . Clearly,

$$\|\gamma^*\lambda\|_{Th} = \|\lambda\|_{Th}$$

holds for any measured lamination λ , and hence $\text{Möb}(\mathbb{D})$ acts by isometry on $\mathcal{ML}_b(\mathbb{D})$.

2.3. Boxes and the Liouville measure. The *cross ratio* of a quadruple (a, b, c, d) is given by $cr(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$. A *box of geodesics* Q in \mathcal{G} is the quotient under the equivalence \sim of the product $[a, b] \times [c, d]$ of two disjoint closed arcs in S^1 , where $[a, b]$ (resp. $[c, d]$) is the arc in S^1 from a (resp. c) to b (resp. d) for the orientation of S^1 . We will write somewhat incorrectly $Q = [a, b] \times [c, d]$ instead of a more correct $Q = ([a, b] \times [c, d]) / \sim$. The *Liouville measure* L is a canonical, non-trivial, Möbius group invariant Borel measure on \mathcal{G} defined by

$$L(Q) = |\log |cr(a, b, c, d)|| = \left| \log \left| \frac{(a-c)(b-d)}{(a-d)(b-c)} \right| \right|$$

for all boxes $Q = [a, b] \times [c, d]$. The Liouville measure is unique up to scaling. The infinitesimal form of the Liouville measure on $\mathcal{G} = (S^1 \times S^1 \setminus \text{diag}) \sim$ is given by (see [1])

$$dL = \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2}.$$

For instance, when we consider the upper half-plane model \mathbb{H} of the hyperbolic plane instead of \mathbb{D} and let $Q = [-1, 1] \times [e^D, -e^D]$, the Liouville measure of Q is

$$(2.1) \quad L(Q) = -2 \log \tanh \frac{D}{2}.$$

Thus, for a general square $Q = [a, b] \times [c, d]$, the Liouville measure $L(Q)$ is inversely related to the hyperbolic distance between the geodesics $[a, b]$ and $[c, d]$. Furthermore, a square $Q = [a, b] \times [c, d]$ satisfies $L(Q) = \log 2$ if and only if the distance D

between $[a, b]$ and $[c, d]$ satisfies $e^D = \omega_0 (= (1 + \sqrt{2})^2)$ if and only if the distance between $[a, b]$ and $[c, d]$ equals the distance between $[a, d]$ and $[b, c]$. A short computation shows that the box $Q = [-1, 1] \times [3 + 2\sqrt{2}, -(3 + \sqrt{2})] \subset (\hat{\mathbb{R}} \times \hat{\mathbb{R}} \setminus \text{diag}) \sim$ has the Liouville measure $\log 2$.

We again consider the unit disk model \mathbb{D} of the hyperbolic plane and define $Q^* = [-i, 1] \times [i, 1]$. Let $\ell_{Q^*} = [e^{-\pi/4}, e^{3\pi/4}] \in Q^*$. Let Q be a box with $L(Q) = \log 2$ and γ_Q a Möbius transformation of \mathbb{D} with $\gamma_Q(Q^*) = Q$. The geodesic $\ell_Q := \gamma_Q(\ell_{Q^*})$ is called the *center* of the box Q .

2.4. Hölder distribution. Let d_0 be the angle metric on S^1 with respect to the origin $0 \in \mathbb{D}$. Let d be the metric on \mathcal{G} induced by d_0 as in §2.1. A Hölder continuous function $\varphi : \mathcal{G} \rightarrow \mathbb{R}$ with respect to the fixed metric d on \mathcal{G} is Hölder continuous for the whole class of Hölder equivalent metrics to the metric d . Unless otherwise stated, all the constructions that follow are with respect to the fixed metric d on \mathcal{G} .

The space Höl_0 consists of all Hölder continuous function $\varphi : \mathcal{G} \rightarrow \mathbb{R}$ with compact support, where \mathcal{G} is equipped with the fixed metric d . Let $0 < \nu \leq 1$. For a ν -Hölder continuous function φ on \mathcal{G} , we define its ν -norm by

$$\|\varphi\|_\nu = \max \left\{ \max |\varphi([x, y])|, \sup \frac{|\varphi([x, y]) - \varphi([x_1, y_1])|}{d([x, y], [x_1, y_1])^\nu} \right\},$$

where the maximum inside the brackets is over all $[x, y] \in \mathcal{G}$ and where the supremum is over all distinct $[x, y], [x_1, y_1] \in \mathcal{G}$. Let us denote by Höl_0^ν the space all ν -Hölder continuous functions on \mathcal{G} with compact support. Then, $\text{Höl}_0 = \cup_{0 < \nu \leq 1} \text{Höl}_0^\nu$.

A ν -test function is a pair (φ, Q) , where Q is a box of geodesics and φ is a Hölder continuous function such that $L(Q) = \log 2$, $\text{supp}(\varphi) \subset Q$ and $\|\varphi \circ \gamma_Q\|_\nu \leq 1$. Recall that γ_Q is a unique Möbius mapping which maps $Q^* = [-i, 1] \times [i, -1]$ onto Q . We denote by $\text{test}(\nu)$ the set of ν -test functions.

A ν -Hölder distribution is a linear functional W on Höl_0^ν such that

$$\|W\|_\nu := \sup\{|W(\varphi)| \mid (\varphi, Q) \in \text{test}(\nu)\} < \infty.$$

A Hölder distribution is a linear functional W on Höl_0 such that

$$\|W\|_\nu < \infty$$

for all $0 < \nu \leq 1$. In general, the ν -Hölder norms $\|W\|_\nu$ of a fixed Hölder distribution W can increase without a bound as $\nu \rightarrow 0$. Let \mathcal{H}^ν be the set of all linear functionals W on Höl_0 with $\|W\|_\nu < \infty$. Then \mathcal{H}^ν is a Banach space for the ν -norm $\|\cdot\|_\nu$. The space \mathcal{H} of all Hölder distributions is equal to $\cap_{0 < \nu \leq 1} \mathcal{H}^\nu$. Each $\|\cdot\|_\nu$ is a norm (i.e. is non-degenerate) on \mathcal{H} , but $(\mathcal{H}, \|\cdot\|_\nu)$ is not a complete space. The family of ν -norms makes \mathcal{H} into a Fréchet space. Note that \mathcal{H} is invariant under quasisymmetric changes of coordinates on S^1 because quasisymmetric maps are Hölder continuous, while each \mathcal{H}^ν is not invariant. For more details, see [13].

Special Test Functions. For the later use, we shall define a special test function $(\psi_{0,\nu}, Q^*)$ ($0 < \nu \leq 1$) as follows. Let

$$(2.2) \quad Q_0^* = [\omega_1^{13}, \omega_1^{15}] \times [\omega_1^5, \omega_1^7]$$

where $\omega_1 = e^{i\pi/8}$ is a 16-th root of unity. We now fix a C^∞ function φ_0 on \mathcal{G} with the properties that $\varphi_0 \equiv 1$ on Q_0^* , $0 \leq \varphi_0 \leq 1$ and $\text{supp}(\varphi_0) \subset Q^*$. Since φ_0 is a Lipschitz function and

$$\|\varphi_0\|_\nu \leq (\pi/2)^{1-\nu} \|\varphi_0\|_1$$

for all ν with $0 < \nu \leq 1$ (cf. the equation (8) in [13]), we have

$$(2.3) \quad (\psi_{0,\nu}, Q^*) := (((\pi/2)^{1-\nu} \|\varphi_0\|_1)^{-1} \varphi_0, Q^*) \in \text{test}(\nu).$$

Notice that $L(Q_0^*) = \log(4/(\sqrt{2} + 2))$.

2.5. Bounded measured laminations as Hölder distributions. A *Radon measure* on a topological space is a locally finite Borel measure with the inner regularity. It is known that any locally finite Borel measure on a Suslin space (for instance, a separable and complete metrizable space) is a Radon measure (cf. Theorem 11 of Chapter II in [18]).

2.5.1. Weak* convergence. We say that a sequence $\{\lambda_n\}_{n=1}^\infty$ of Borel measures on \mathcal{G} converges in the weak* topology to a Borel measure λ if for all continuous function f with compact support on \mathcal{G} , it holds

$$\lim_{n \rightarrow \infty} \int_{\mathcal{G}} f d\lambda_n = \int_{\mathcal{G}} f d\lambda.$$

(This convergence is sometimes called the *vague convergence*, but we call it the weak* convergence here.)

2.5.2. Measures of squares. The following lemma is well-known. However we give a proof for readers convenience.

Lemma 2.1 (Comparison with Thurston norm). *There is a universal constant C_0 such that for any measured lamination λ , we have*

$$\frac{1}{C_0} \|\lambda\|_{Th} \leq \sup_Q \lambda(Q) \leq \|\lambda\|_{Th},$$

where the supremum is taken over all boxes Q with $L(Q) = \log 2$.

Proof. Let I be a geodesic arc in \mathbb{D} of the unit length which intersects transversely a leaf ℓ of λ . Since the support $|\lambda|$ consists of disjoint geodesics, there is a universal constant L_0 with the following property: Let J be a geodesic arc in \mathbb{D} of length L_0 which is orthogonal to ℓ at the midpoint of J and let the midpoint of J be equal to $I \cap \ell$. Then, any leaf of $|\lambda|$ with non-trivial intersection with I also intersects J .

One can check that any leaf of $|\lambda|$ ($\subset \mathcal{G}$) which intersects J is contained in a box Q' with center ℓ satisfying $L(Q') = 2 \log \cosh(L_0/2)$. To see this, we identify \mathbb{D} with the upper half-plane \mathbb{H} and normalize J and ℓ such that $J = [1, e^{L_0}]i$ and $\ell = \{|z| = e^{L_0/2}\} \cap \mathbb{H}$. Any complete geodesic which is disjoint from ℓ and which intersects J is in the box $Q' = [e^{3L_0/2}, -e^{L_0/2}] \times [e^{L_0/2}, e^{3L_0/2}]$. This means that $\lambda(I) \leq \lambda(J) \leq \lambda(Q')$ and hence we conclude

$$\|\lambda\|_{Th} \leq C_0 \sup_Q \lambda(Q)$$

with universal constant $C_0 > 0$, where the supremum runs over all boxes Q with $L(Q) = \log 2$.

To show the converse, let $Q = [a, b] \times [c, d]$ be a box in \mathcal{G} . The measure $\lambda(Q)$ is obtained as follows. Suppose for the simplicity that a, b, c and d are lying on S^1 in this order. Let $\ell_1 = [a, d]$ and $\ell_2 = [b, c]$ and I the geodesic segment which intersects orthogonally to ℓ_1 and ℓ_2 at endpoints. Then, any complete geodesic in

Q intersects I . Since the length of I is $\log 2 < 1$, there is a geodesic arc I' of unit length which contains I and hence we obtain

$$\lambda(Q) \leq \lambda(I') \leq \|\lambda\|_{Th},$$

for all boxes Q with $L(Q) = \log 2$ which implies the desired inequality. \square

2.6. Hölder distributions defined from measures. Any $\lambda \in \mathcal{ML}_b(\mathbb{D})$ induces a Hölder distribution by the formula

$$\text{Höl}_0 \ni \varphi \mapsto \int_{\mathcal{G}} \varphi d\lambda.$$

Indeed, by definition and Lemma 2.1, we have

$$\|\lambda\|_{\nu} = \sup_{(\varphi, Q) \in \text{test}(\nu)} \left| \int_Q \varphi d\lambda \right| \leq \sup_Q \lambda(Q) \leq \|\lambda\|_{Th}$$

for all $0 < \nu \leq 1$, where in the third term, Q runs over all boxes Q with $L(Q) = \log 2$. Thus the above formula gives a natural inclusion of $\mathcal{ML}_b(\mathbb{D})$ into \mathcal{H} .

The following lemma extends the above equivalence of norms to any locally finite Borel measure on \mathcal{G} .

Lemma 2.2. *Let λ be a locally finite Borel measure on \mathcal{G} . Then the induced linear functional*

$$(2.4) \quad \lambda : \text{Höl}_0 \ni \varphi \mapsto \int_{\mathcal{G}} \varphi d\lambda$$

is a Hölder distribution if and only if $\sup_Q \lambda(Q) < \infty$, where the supremum is over all boxes Q with $L(Q) = \log 2$. In this case, there is a universal constant $C_1 > 0$ such that

$$\|\lambda\|_{\nu} \leq \sup_Q \lambda(Q) \leq C_1 \|\lambda\|_{\nu},$$

for all ν with $0 < \nu \leq 1$, where $L(Q) = \log 2$, and $\|\lambda\|_{\nu}$ is the ν -norm of the Hölder distribution (2.4).

Proof. From (2.3), we obtain

$$\lambda(Q_0^*) \leq \int_{Q^*} \varphi_0 d\lambda \leq ((\pi/2)^{1-\nu} \|\varphi_0\|_1) \|\lambda\|_{\nu} \leq C'_1 \|\lambda\|_{\nu},$$

where C'_1 is a universal constant. Since Q^* is covered by finitely many boxes which are the images of Q_0^* under Möbius transformations, by applying the argument above to $(\gamma_Q)^* \lambda$ and $\varphi_0 \circ \gamma_Q^{-1}$ instead of λ and φ_0 , we conclude that

$$\lambda(Q) \leq C_1 \|\lambda\|_{\nu}.$$

for all Q with $L(Q) = \log 2$, where C_1 is a universal constant. The left-hand side follows from the standard argument. Indeed, since $\|\varphi \circ \gamma_Q\|_{\nu} \leq 1$, $\sup_Q |\varphi| \leq 1$ and hence for any $\epsilon > 0$, we can take $(\varphi, Q) \in \text{test}(\nu)$ such that

$$\|\lambda\|_{\nu} \leq \left| \int_Q \varphi d\lambda \right| + \epsilon \leq \lambda(Q) + \epsilon \leq \sup_Q \lambda(Q) + \epsilon,$$

which implies what we wanted. \square

3. EARTHQUAKES AND EARTHQUAKE MEASURES

3.1. Earthquakes. Let \mathcal{L} be a geodesic lamination in \mathbb{D} . An *earthquake* E with the support \mathcal{L} is a surjective map $E : \mathbb{D} \rightarrow \mathbb{D}$ such that E is a hyperbolic isometry when restricted to any stratum of \mathcal{L} and, for any two strata A and B , the *comparison isometry*

$$\text{cmp}(A, B) = (E|_A)^{-1} \circ E|_B$$

is a hyperbolic translation whose axis weakly separates A and B , and which translates B to the left as seen from A . An earthquake E of \mathbb{D} continuously extends to a homeomorphism of the boundary S^1 (see [20]). We denote by $E|_{S^1}$ the extension.

Given an earthquake E with support \mathcal{L} , there is an associated positive transverse measure λ to \mathcal{L} as follows. Let I be a closed geodesic arc transversely intersecting \mathcal{L} with arbitrary orientation. For given n , choose a closed geodesic arc I_n which contains I in its interior such that $I_{n+1} \subsetneq I_n$ and $\cap_n I_n = I$. Furthermore, choose strata $\mathcal{A}_n = \{A_0, A_1, \dots, A_{k(n)}, A_{k(n)+1}\}$ of the support of E such that A_0 contains the left end point of I_n , A_1 contains the left endpoint of I , $A_{k(n)}$ contains the right endpoint of I , $A_{k(n)+1}$ contains the right endpoint of I_n , A_i 's intersect I in the given order and the maximum of the distances between the consecutive intersections of \mathcal{A}_n with I_n goes to zero as $n \rightarrow \infty$. The summation of the translation lengths of the comparison isometries $\text{cmp}(A_i, A_{i+1}) = (E|_{A_i})^{-1} \circ E|_{A_{i+1}}$ for $i = 0, 1, \dots, k(n)+1$ is the approximate measure of I . If $n \rightarrow \infty$ and \mathcal{A}_n are chosen such that $(\cup_{i=1}^{k(n)} A_i) \cap I$ is dense in I for all n , the limit of approximate measure is a well-defined positive finite Borel measure ([20] and [7]). (Note that if $E : \mathbb{D} \rightarrow \mathbb{D}$ is continuous at the endpoints of I then we can replace I_n with I for each n in the above construction.) This transverse measure defines a measured lamination λ with support \mathcal{L} . We call the measured lamination λ the *earthquake measure* for E . We denote by E^λ a earthquake map with earthquake measure λ . An earthquake map is (essentially) uniquely determined by its earthquake measure. The ambiguity is up to postcomposition of the earthquake map by a Möbius map and on each leaf where the earthquake has a discontinuity there is a range of possibilities (but the extension to S^1 gives the same map regardless of the choices in this range.) The set of strata where an earthquake map has a discontinuity consists of at most countable family of leaves of \mathcal{L} .

In [20], Thurston showed that for any orientation preserving homeomorphism h on $\partial\mathbb{D}$, there is a unique earthquake map E^λ such that $h = E^\lambda|_{S^1}$. Thurston's theorem induces an injective map from the space of right cosets of $\text{Möb}(\mathbb{D})$ in the group of orientation preserving homeomorphisms into the space of measured laminations in \mathbb{D} by the formula $\text{Möb}(\mathbb{D}) \circ h \mapsto \lambda$ where $h = E^\lambda|_{S^1}$.

For an orientation preserving homeomorphism $h : S^1 \rightarrow S^1$ and the earthquake map $E^\lambda|_{S^1} = h$, we have that $h \circ \gamma = E^{\gamma^*(\lambda)}|_{S^1}$ for any $\gamma \in \text{Möb}(\mathbb{D})$.

3.2. Convergence of earthquakes. Notice from the definition that for any $\gamma \in \text{Möb}(\mathbb{D})$, the earthquake measure of $\gamma \circ E$ coincide with that of E . Hence, E^λ is determined up to postcomposition of Möbius transformations. Because of this ambiguity, we should give a remark on the symbol E^λ . Namely, when E^λ is treated as a map, this E^λ is always chosen suitably for the content. For instance, we have used the equation " $h = E^\lambda$ " with a homeomorphism h on S^1 .

This equation means that we can choose an earthquake map with earthquake measure λ which coincides with h on S^1 . When we say that " $E^{\lambda_n} \rightarrow E^\lambda$ as $n \rightarrow \infty$ ",

a sequence consisting of choices of the earthquake maps for λ_n ($n \in \mathbb{N}$) converges to one of those for λ .

4. THE UNIVERSAL TEICHMÜLLER SPACE AND THE EARTHQUAKE MEASURE MAP

4.1. Quasisymmetric maps. An orientation preserving homeomorphism h is said to be a *quasisymmetric* if there is a constant $M \geq 1$ such that

$$(4.1) \quad \frac{1}{M} \leq \frac{|h(J_1)|}{|h(J_2)|} \leq M$$

for all adjacent intervals $J_1, J_2 \subset S^1$ with $|J_1| = |J_2|$, where $|J_i|$ is the arc length with respect to the angle measure on $S^1 = \partial\mathbb{D}$. Let \mathcal{QS} be the set of all quasisymmetric maps on S^1 . The *universal Teichmüller space* $T(\mathbb{D})$ is the quotient space

$$T(\mathbb{D}) = \text{Möb}(\mathbb{D}) \backslash \mathcal{QS}$$

where the group $\text{Möb}(\mathbb{D})$ of Möbius transformations acts on \mathcal{QS} via post-compositions. For any $h \in \mathcal{QS}$, we denote by $[h]$ its class in $T(\mathbb{D})$. The universal Teichmüller space $T(\mathbb{D})$ admits a natural (metric) topology inherited from the maximal dilatations. Namely, two quasisymmetric maps h_1 and h_2 are close if there exists a quasiconformal extension of $h_2 \circ h_1^{-1}$ whose maximal dilatation is near one. This topology on $T(\mathbb{D})$ is the same one inherited from quasisymmetric constants. See [3] or [6].

4.2. The earthquake measure map. In this subsection, we define the earthquake measure map. We first recall the following theorem, which is proved by Gardiner-Hu-Lakic [7] and in [14].

Theorem 4 (Gardiner-Hu-Lakic, Šarić). *Let h be an orientation preserving homeomorphism h of $\partial\mathbb{D} = S^1$ and let E^λ be the earthquake of \mathbb{D} whose continuous extension to S^1 equals h . Then the following are equivalent.*

- (1) *The earthquake measure λ of the earthquake $E^\lambda|_{S^1} = h$ is bounded.*
- (2) *h is quasisymmetric.*

The earthquake measure map

$$\mathcal{EM} : T(\mathbb{D}) \rightarrow \mathcal{ML}_b(\mathbb{D})$$

is defined by $\mathcal{EM}([h]) = \lambda$ where $h = E^\lambda|_{S^1}$. As noted in §3.2, every earthquake is determined by its earthquake measure up to post-composition by Möbius maps. Hence, together with the uniqueness of the earthquake measures for homeomorphisms [20], Theorem 4 tells us that the earthquake measure map \mathcal{EM} is well-defined and bijective.

In [7] and [8], it is proved that for a quasisymmetric map h , the Thurston norm of the earthquake measure of h is comparable with the quasisymmetric constant of h . We will give a brief proof of a weaker result than the comparison statement which we need here (cf. Lemma 6.2).

5. AN EXAMPLE

In this section, we consider the example from Introduction of non-convergence of a sequence in the space of bounded measured laminations in the Fréchet topology which converges in the weak* topology.

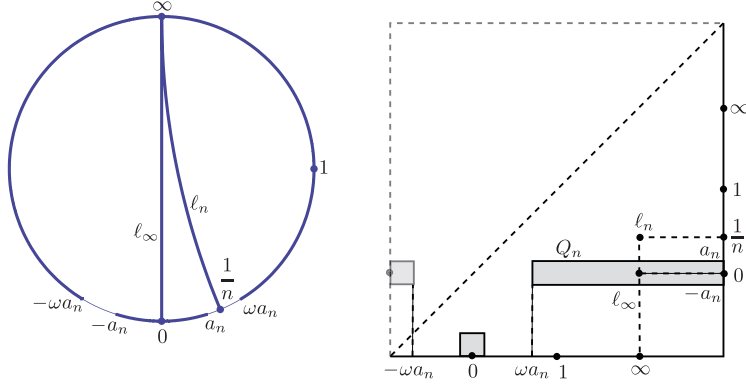


FIGURE 2. ℓ_∞ , ℓ_n , and the box Q_n with center ℓ_∞ and $L(Q_n) = \log 2$ such that $\ell_n \notin Q_n$. The right picture represents how Q_n distributes in the space \mathcal{G} .

5.1. Fréchet topology vs weak* topology. For the simplicity, we use the upper half-plane model \mathbb{H} for the hyperbolic plane in place of \mathbb{D} . Let $\ell_n = [1/n, \infty]$ ($n \in \mathbb{Z} \setminus \{0\}$) and $\ell_\infty = [0, \infty]$ in \mathcal{G} .

Example 1. Let λ_n be the measured lamination whose support is ℓ_n with $\lambda_n(\ell_n) = 1$. Let λ_∞ be the measured lamination whose support is ℓ_∞ such that $\lambda_\infty(\ell_\infty) = 1$. Then λ_n does *not* converge to λ_∞ in the Fréchet topology as $n \rightarrow \infty$, while it does converge in the weak* topology on measures on \mathcal{G} .

Indeed, for $n \geq 1$ and $\omega_0 = (1 + 2\sqrt{2})^2$, we define a box $Q_n = [-a_n, a_n] \times [\omega_0 a_n, -\omega_0 a_n]$ with $1/(\omega_0 n) < a_n < 1/n$, where $[\omega_0 a_n, -\omega_0 a_n]$ is the interval in $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ which contains ∞ and connects $\omega_0 a_n$ and $-\omega_0 a_n$ (cf. Figure 2).

Then, one can check that $L(Q_n) = \log 2$, $\lambda_\infty(Q_n) = 1$ and $\lambda_n(Q_n) = 0$ since $\ell_n \notin Q_n$. We take a Lipschitz function on \mathcal{G} with support in Q^* such that $\|\varphi\|_1 \leq 1$ and the value at the center ℓ_{Q^*} of φ is positive. Set $\varphi_{\nu,n} = (2/\pi)^{1-\nu} \varphi \circ (\gamma_{Q_n})^{-1}$ for $0 < \nu \leq 1$. From the symmetries of Q_n and Q^* , one can see that $\gamma_{Q_n}(\ell_\infty) = \ell_{Q_n}$ for all n . Thus, by (2.4), the pair $(\varphi_{\nu,n}, Q_n)$ is in $\text{test}(\nu)$ and satisfies

$$(5.1) \quad \|\lambda_n - \lambda_\infty\|_\nu \geq \left| \int_{Q_n} \varphi_{\nu,n} d(\lambda_n - \lambda_\infty) \right| = (2/\pi)^{1-\nu} \varphi_{\nu,n}(\ell_\infty) \geq (2/\pi) \varphi(\ell_{Q^*})$$

for all n and $0 < \nu \leq 1$, which implies what we wanted. By the same reason, we can see that the “midpoint approximation” $\frac{1}{2}(\lambda_n + \lambda_{-n})$ does not converge to λ_∞ in the Fréchet topology either. We generalize this example in the following proposition.

Proposition 5.1. *Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of bounded measured laminations which converges in the Fréchet topology to a measured lamination λ_∞ whose support is a single geodesic. Then, for all sufficiently large n , each endpoint of $|\lambda_\infty|$ is contained in the closure the set of endpoints of leaves of λ_n .*

Proof. Let $|\lambda_\infty| = [0, \infty]$. Suppose on the contrary that there is a $\delta_n > 0$ such that any leaf of λ_n does not have endpoints in an open interval $(-\delta_n, \delta_n)$. We take a sufficiently small $a_n > 0$ such that $\omega_0 a_n < \delta_n$, where $\omega_0 = (1 + \sqrt{2})^2$ as before. Define Q_n by

$$Q_n = [-a_n, a_n] \times [\omega_0 a_n, -\omega_0 a_n]$$

Then, the center of Q_n is ℓ_∞ , $L(Q_n) = \log 2$ and $Q_n \cap |\lambda_n| = \emptyset$. Thus, by the same calculation as (5.1), we get

$$\|\lambda_n - \lambda_\infty\|_\nu \geq (2/\pi)\varphi(\ell_\infty)$$

for some Lipschitz function φ independent of ν . This means that $\{\lambda_n\}_{n=1}^\infty$ can not converge to λ_∞ in the Fréchet topology. \square

Unfortunately, Proposition 5.1 does not give a characterization of bounded measured laminations in a neighborhood of an elementary measured lamination which is illustrated by Example 1.

5.2. Elementary Earthquakes. We shall check the behavior of earthquakes whose supports are single geodesics given in the above section to clarify the connection between the Fréchet topology and the weak* topology on the measured laminations and the Teichmüller topology on the extensions to S^1 of their corresponding earthquake maps.

Let $\ell_n = [1/n, \infty]$ for $n \in \mathbb{N} \cup \{\infty\}$. Then the earthquake map E^{λ_n} for elementary measures λ_n with single geodesic support ℓ_n and mass 1 (normalized to fix three points $\{-1, 0, \infty\}$) is

$$E^{\lambda_n}(z) = \begin{cases} e(z - 1/n) + 1/n & (\operatorname{Re}(z) > 1/n) \\ z & (\operatorname{Re}(z) \leq 1/n) \end{cases}$$

for $z \in \mathbb{H}$, where we set $1/\infty = 0$. Clearly $h_n := E^{\lambda_n}|_{\partial\mathbb{H}}$ converges to $h_\infty = E^{\lambda_\infty}|_{\partial\mathbb{H}}$ pointwise. However, h_n does not converge to h_∞ in the Teichmüller topology. Indeed, for $n \in \mathbb{N}$ and boxes $Q_n = [\infty, -e/n] \times [0, e/n]$, we get $L(Q_n) = \log 2$ and

$$L(h_n \circ h_\infty^{-1}(Q_n)) = \log(e + 1) - 1.$$

This means that the maximal dilatation of any quasiconformal extension of $h_n \circ h_\infty^{-1}$ is uniformly greater than 1. Thus, a sequence $\{h_n\}_{n=1}^\infty$ does not converge to h_∞ in $T(\mathbb{H})$, which also follows from Theorem 1 and Example 1 above.

6. THE EARTHQUAKE MEASURE MAP IS A HOMEOMORPHISM

In this section, we prove Theorem 1. To do so, we define a *uniform-weak* topology* on $\mathcal{ML}_b(\mathbb{D})$ (see [15]) and show that it is equivalent to the restriction of the Fréchet topology.

6.1. Uniform-weak* topology. We say that a sequence $\lambda_m \in \mathcal{ML}_b(\mathbb{D})$ converges to $\lambda \in \mathcal{ML}_b(\mathbb{D})$ in the *uniform-weak* topology* if for any continuous function f on \mathcal{G} with $\operatorname{supp}(f) \subset Q^*$,

$$\sup_Q \int_{Q^*} f d((\gamma_Q)^*(\lambda_m) - (\gamma_Q)^*(\lambda)) \rightarrow 0$$

as $m \rightarrow \infty$, where the supremum is over all boxes Q with $L(Q) = \log 2$ and $\gamma_Q \in \operatorname{Möb}(\mathbb{D})$ is such that $\gamma_Q(Q^*) = Q$.

6.2. Two lemmas. Let us start with the following lemma.

Lemma 6.1. *Let $\{\lambda_m\}_{m \in \mathbb{N}}$ be a sequence of bounded measured laminations and λ a bounded measured lamination. Suppose that there exists $C > 0$ such that $\|\lambda_m\|_{Th} < C$ for all $m \in \mathbb{N}$. Then, the following are equivalent.*

- (1) *The sequence $\{\lambda_m\}_m$ converges to $\lambda \in \mathcal{ML}_b(\mathbb{D})$ in the uniform-weak* topology.*
- (2) *The sequence $\{\lambda_m\}_m$ converges to $\lambda \in \mathcal{ML}_b(\mathbb{D})$ in the Fréchet topology.*

Proof. Assume that (1) holds. Seeking a contradiction, we suppose (2) does not hold. Then, by taking a subsequence of $\{\lambda_m\}_m$ if necessary, there are $\nu, \epsilon_0 > 0$ and a sequence $\{(\varphi_m, Q_m)\}_{m=1}^\infty$ in $\text{test}(\nu)$ such that

$$(6.1) \quad \left| \int_{Q^*} \varphi_m \circ \gamma_{Q_m} d\hat{\lambda}_m \right| = \left| \int_{Q_m} \varphi_m d(\lambda_m - \lambda) \right| \geq \epsilon_0.$$

for all m , where we set $\hat{\lambda}_m = (\gamma_{Q_m}^* \lambda_m) - (\gamma_{Q_m}^* \lambda)$ for the simplicity. From the definition of a test function, $\varphi_m \circ \gamma_{Q_m}$ satisfies $\|\varphi_m \circ \gamma_{Q_m}\|_\nu \leq 1$. Hence, by Ascoli-Arzelà's theorem, the sequence contains a convergence subsequence $\{\varphi_{m_j} \circ \gamma_{Q_{m_j}}\}_j$ in the C^0 -topology. We denote by ψ_∞ its limit.

Since the support of $\varphi_{m_j} \circ \gamma_{Q_{m_j}}$ is contained in Q^* , so is that of ψ_∞ . Notice that

$$(6.2) \quad \int_{Q^*} \psi_\infty d\hat{\lambda}_{m_j} = \int_{Q^*} \varphi_{m_j} \circ \gamma_{Q_{m_j}} d\hat{\lambda}_{m_j} + \int_{Q^*} (\psi_\infty - \varphi_{m_j} \circ \gamma_{Q_{m_j}}) d\hat{\lambda}_{m_j}.$$

Since the Thurston norm of λ_{m_j} is uniformly bounded, it follows that the last term of the right-hand side of (6.2) tends to zero. From (6.1), we get

$$\sup_{Q, L(Q)=\log 2} \left| \int_{Q^*} \psi_\infty d((\gamma_Q^* \lambda_{m_j}) - (\gamma_Q^* \lambda)) \right| \geq \left| \int_{Q^*} \psi_\infty d\hat{\lambda}_{m_j} \right| \geq \epsilon_0/2$$

for sufficiently large j , which contradicts (1). Thus (1) implies (2).

We now assume that (2) holds, and that (1) does not hold and seek a contradiction again. Then, after taking a subsequence of $\{\lambda_m\}_{m=1}^\infty$ if necessary, there exist $\epsilon_0 > 0$ and a continuous function f on \mathcal{G} with $\text{supp}(f) \subset Q^*$ such that

$$\sup_Q \left| \int_{Q^*} f d((\gamma_Q)^*(\lambda_m) - (\gamma_Q)^*(\lambda)) \right| \geq 2\epsilon_0$$

for all m , where the supremum is taken over all squares Q with $L(Q) = \log 2$. This implies that there is a sequence $\{Q_m\}_{m=1}^\infty$ of boxes such that $L(Q_m) = \log 2$ and

$$(6.3) \quad \left| \int_{Q^*} f d\hat{\lambda}_m \right| \geq \epsilon_0$$

for $m \geq 1$, where we set $\hat{\lambda}_m = (\gamma_{Q_m})^*(\lambda_m) - (\gamma_{Q_m})^*(\lambda)$.

Let $\epsilon > 0$. Take a ν -Hölder function φ_ϵ with $\text{supp}(\varphi_\epsilon) \subset Q^*$ such that the supremum norm of $f - \varphi_\epsilon$ is less than ϵ . Let $\psi_m = (\|\varphi_\epsilon\|_\nu)^{-1}(\varphi_\epsilon \circ \gamma_{Q_m}^{-1})$. Then, a pair (ψ_m, Q_m) is in $\text{test}(\nu)$ and it satisfies

$$(6.4) \quad \begin{aligned} \int_{Q_m} \psi_m d(\lambda_m - \lambda) &= \frac{1}{\|\varphi_\epsilon\|_\nu} \int_{Q^*} \varphi_\epsilon d\hat{\lambda}_m \\ &= \frac{1}{\|\varphi_\epsilon\|_\nu} \left(\int_{Q^*} f d\hat{\lambda}_m + \int_{Q^*} (\varphi_\epsilon - f) d\hat{\lambda}_m \right). \end{aligned}$$

By Lemma 2.1 and by our assumption that Thurston norms of λ_m are uniformly bounded, the last term in the parentheses of (6.4) is less than $C_1\epsilon$ for some $C_1 > 0$ independent of m and ϵ (and hence ν). By (6.3), we get

$$\left| \int_{Q_m} \psi_m d(\lambda_m - \lambda) \right| \geq \frac{1}{\|\varphi_\epsilon\|_\nu} (\epsilon_0 - C_1\epsilon).$$

Hence, if we take $\epsilon > 0$ (and $\nu > 0$) so that $C_1\epsilon < \epsilon_0/2$, we obtain

$$\sup_{(\varphi, Q) \in \text{test}(\nu)} \left| \int_Q \varphi d(\lambda_m - \lambda) \right| \geq \left| \int_{Q_m} \psi_m d(\lambda_m - \lambda) \right| \geq \frac{\epsilon_0}{2\|\varphi_\epsilon\|_\nu}$$

for all m , which contradicts (2), since the constant on the right-hand side is independent of m . Thus (2) implies (1). \square

We need the following lemma.

Lemma 6.2. *For any $C_1 > 0$, there is $C_2 > 0$ depending only of C_1 such that for any bounded measured lamination λ with $\|\lambda\|_\nu \leq C_1$ for some ν with $0 < \nu \leq 1$, the quasisymmetric constant of $E^\lambda|_{S^1}$ is at most C_2 .*

Proof. This follows from the results in [12]. Indeed, Lemma 2.2 implies that $\|\lambda\|_{Th} < \infty$. Then the earthquake path $t \mapsto E^{t\lambda}|_{S^1}$ is a real analytic path in the universal Teichmüller space $T(\mathbb{D})$ which extends to a holomorphic motion $\tau \mapsto E^{\tau\lambda}|_{S^1}$ of S^1 in $\hat{\mathbb{C}}$. Moreover, the holomorphic motion is well-defined for τ in a neighborhood of the real line \mathbb{R} whose shape depends only on $\|\lambda\|_{Th}$ (see [12]). Then the essential supremum norm of the Beltrami coefficient of the extension of the holomorphic motion of S^1 to a holomorphic motion of $\hat{\mathbb{C}}$ for $\tau = 1$ depends only on the shape of the domain in which τ is defined. As we noted above, this in turn only depends on $\|\lambda\|_{Th}$. Thus the quasisymmetric constant of $E^\lambda|_{S^1}$ depends only on $\|\lambda\|_{Th}$ which proves the lemma. An alternative proof would use results in [7] or in [8]. \square

6.3. Proof of Theorem 1. We first show that the earthquake measure map \mathcal{EM} is continuous. Let $[h] \in T(\mathbb{D})$ and $\{[h_m]\}_{m=1}^\infty \subset T(\mathbb{D})$ with $[h_m] \rightarrow [h]$ as $m \rightarrow \infty$. Let $\lambda_m = \mathcal{EM}([h_m])$ and $\lambda = \mathcal{EM}([h])$. Then, it follows from Lemma 4.1 of [15] that for any continuous function f on \mathcal{G} with $\text{supp}(f) \subset Q^*$,

$$\sup_Q \int_{Q^*} f d((\gamma_Q)^*(\lambda_m) - (\gamma_Q)^*(\lambda)) \rightarrow 0$$

as $m \rightarrow \infty$, where Q runs over all boxes whose Liouville measures are $\log 2$. Hence, by Lemma 6.1, we have

$$\|\lambda_n - \lambda\|_\nu = \sup_{(\varphi, Q) \in \text{test}(\nu)} \left| \int_Q \varphi d(\lambda_m - \lambda) \right| \rightarrow 0$$

as $m \rightarrow \infty$, for all ν . This means that \mathcal{EM} is continuous.

Next, we show that the inverse \mathcal{EM}^{-1} is continuous. Suppose $\lambda_n = \mathcal{EM}([h_m]) \rightarrow \lambda = \mathcal{EM}([h])$. Assume on the contrary that \mathcal{EM}^{-1} is not continuous. Namely, there are $\epsilon_0 > 0$ and a sequence $\{Q_m\}_{m=1}^\infty$ of boxes with $L(Q_m) = \log 2$ such that

$$(6.5) \quad |L(h_m(Q_m)) - L(h(Q_m))| \geq \epsilon_0$$

for all m , where h and h_m are normalized to fix 1, i and -1 . Take Möbius transformations β_m and β_m^* such that $g_m = \beta_m \circ h_m \circ \gamma_{Q_m}$ and $g_m^* = \beta_m^* \circ h \circ \gamma_{Q_m}$ fix 1, i and -1 . By (6.5), we have

$$(6.6) \quad |L(g_m(Q^*)) - L(g_m^*(Q^*))| \geq \epsilon_0$$

for all m . Since $\lambda_n \rightarrow \lambda$ in the Fréchet topology, it follows that $\|\lambda_n\|_\nu$ is uniformly bounded. Lemma 6.2 implies that the constants of quasimetry of g_m and g_m^* are uniformly bounded. The compactness of normalized quasimetric mappings with uniformly bounded quasimetric constants imply that g_m and g_m^* have two subsequences which are indexed by the same set that converge to quasimetric mappings g and g^* , respectively. For simplicity of notation, we rename the subsequences to be g_m and g_m^* . By (6.6), g does not coincide with g^* .

We claim

Claim. The limits, in the weak* topology, of a pair of converging subsequences $\{(\gamma_{Q_{m_j}})^* \lambda_{m_j}\}_{j=1}^\infty$ and $\{(\gamma_{Q_{m_j}})^* \lambda\}_{j=1}^\infty$ of $\{(\gamma_{Q_m})^* \lambda_m\}_{m=1}^\infty$ and $\{(\gamma_{Q_m})^* \lambda\}_{m=1}^\infty$ is the same bounded measured lamination λ' .

Proof of the Claim. From the compactness of probability measures under the weak* topology, one sees that two sequences $\{(\gamma_{Q_m})^* \lambda_m\}_{m=1}^\infty$ and $\{(\gamma_{Q_m})^* \lambda\}_{m=1}^\infty$ contain a pair $\{(\gamma_{Q_{m_j}})^* \lambda_{m_j}\}_{j=1}^\infty$ and $\{(\gamma_{Q_{m_j}})^* \lambda\}_{j=1}^\infty$ of converging subsequences in the weak* topology. Since λ_m converges to λ in the Fréchet topology, by Lemma 6.1, $\{(\gamma_{Q_m})^* \lambda_m - (\gamma_{Q_m})^* \lambda\}_{m=1}^\infty$ converges to zero measure in the weak* sense. Hence the weak* limits of the pair of converging subsequences $\{(\gamma_{Q_{m_j}})^* \lambda_{m_j}\}_{j=1}^\infty$ and $\{(\gamma_{Q_{m_j}})^* \lambda\}_{j=1}^\infty$ are same. \square

We continue the proof of Theorem 1. By Lemma 3.2 of [14], we can choose representatives of earthquakes $E^{(\gamma_{Q_m})^* \lambda_m}$ and $E^{(\gamma_{Q_m})^* \lambda}$ such that the two sequences $\{E^{(\gamma_{Q_m})^* \lambda_m}|_{S^1}\}_{m=1}^\infty$ and $\{E^{(\gamma_{Q_m})^* \lambda}|_{S^1}\}_{m=1}^\infty$ converge to the same (representative of) earthquake map $E^{\lambda'}|_{S^1}$ pointwise on S^1 (cf. §3.2). Then we take Möbius transformations $\hat{\beta}_m$ and $\hat{\beta}_m^*$ such that $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda_m}$ and $\hat{\beta}_m^* \circ E^{(\gamma_{Q_m})^* \lambda}$ fix 1, i and -1 . Since the limits of two sequences $\{E^{(\gamma_{Q_m})^* \lambda_m}\}_{m=1}^\infty$ and $\{E^{(\gamma_{Q_m})^* \lambda}\}_{m=1}^\infty$ are same, $\hat{\beta}_m$ and $\hat{\beta}_m^*$ converge the same Möbius transformation. Hence, the limits of $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda_m}$ and $\hat{\beta}_m^* \circ E^{(\gamma_{Q_m})^* \lambda}$ also agree.

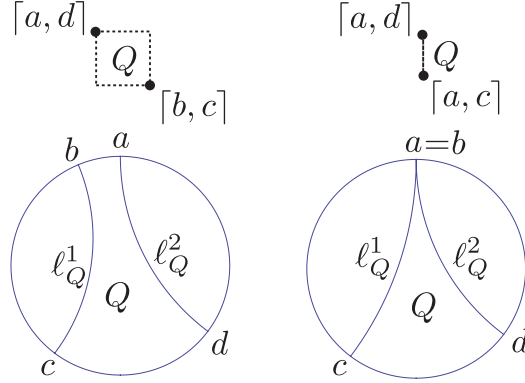
On the other hand, from the definition of earthquakes we have that

$$\begin{aligned} \mathcal{EM}([\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda_m}|_{S^1}]) &= \mathcal{EM}([E^{(\gamma_{Q_m})^* \lambda_m}|_{S^1}]) = (\gamma_{Q_m})^* \lambda_m \\ &= \mathcal{EM}([h_m \circ \gamma_{Q_m}]) = \mathcal{EM}([g_m]) \end{aligned}$$

and

$$\begin{aligned} \mathcal{EM}([\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda}|_{S^1}]) &= \mathcal{EM}([E^{(\gamma_{Q_m})^* \lambda}|_{S^1}]) = (\gamma_{Q_m})^* \lambda \\ &= \mathcal{EM}([h \circ \gamma_{Q_m}]) = \mathcal{EM}([g_m^*]). \end{aligned}$$

Since the earthquake measure map is bijective and all maps $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda_m}$, $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda}$, g_m , and g_m^* fix 1, i and -1 , we conclude $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda_m}|_{S^1} = g_m$ and $\hat{\beta}_m \circ E^{(\gamma_{Q_m})^* \lambda}|_{S^1} = g_m^*$. However, this contradicts that the limits g and g^* of $\{g_m\}_{m=1}^\infty$ and $\{g_m^*\}_{m=1}^\infty$ are distinct. The contradiction proves Theorem 1.

FIGURE 3. Generalized boxes in \mathcal{G} and their extreme geodesics.

7. APPROXIMATIONS BY DISCRETE LAMINATIONS

The purpose of this section is to propose a candidate for a class of *nice* measured laminations in order to better understand the universal Teichmüller space using earthquake maps. Indeed, we will show that *discrete measured laminations* are dense in $\mathcal{ML}_b(\mathbb{D})$ with respect to the Fréchet topology.

7.1. Discrete laminations. A geodesic lamination \mathcal{L} is said to be *discrete* if any compact set $K \subset \mathbb{D}$ intersects only finitely many leaves of \mathcal{L} . Equivalently, \mathcal{L} is a discrete geodesic lamination if it is discrete subset of \mathcal{G} . A measured lamination λ is, by definition, *discrete* if its support $|\lambda|$ is a discrete subset of \mathcal{G} . To show the density of discrete measured laminations in $\mathcal{ML}_b(\mathbb{D})$, we give some notations needed in the proof of the density theorem.

Extreme geodesics and peaks. We recall that a box of geodesics is the product set $I \times J \in \mathcal{G}$ where I and J are disjoint closed intervals of $\partial\mathbb{D} = S^1$. In this proof, we generalize the notion of boxes such that either I or J is allowed to be a point, open or half-open interval. For a generalized box $Q = I \times J$, we define the *extreme geodesics* $\{\ell_Q^1, \ell_Q^2\}$ for Q as follows. Suppose that both I and J are non-degenerate intervals. Let $\text{Int}(I) = (a, b)$ and $\text{Int}(J) = (c, d)$. Then, we set $\ell_Q^1 = [a, d]$ and $\ell_Q^2 = [b, c]$. When exactly one of the intervals is degenerate, say when $I = \{a\}$ and $\text{Int}(J) = (c, d)$, we set $\ell_Q^1 = [a, d]$ and $\ell_Q^2 = [a, c]$. When I and J are both degenerate, ℓ_Q^1 and ℓ_Q^2 are defined to be the geodesic connecting I and J . See Figure 3.

Let $Q = I \times J$ be a generalized box in \mathcal{G} and \mathcal{L} a geodesic lamination. Let $\bar{Q} = \bar{I} \times \bar{J}$ be the closure of Q , where \bar{I}, \bar{J} are closures of I, J . A leaf g of \mathcal{L} is said to be *peak with respect to Q* if $g \in \bar{Q}$ and one of the two components of $\mathbb{D} \setminus g$ does not contain leaves of $\mathcal{L} \cap Q$. By definition, when $\mathcal{L} \cap \bar{Q}$ contains at least two leaves, there is exactly two peak geodesics of \mathcal{L} with respect to Q . In addition, if an extreme geodesic of Q is a leaf of \mathcal{L} , it is also a peak geodesic of \mathcal{L} with respect to Q .

7.2. Density of discrete laminations. We are ready to prove the density of discrete laminations.

Theorem 5 (Discrete laminations are dense). *The set of discrete bounded measured laminations is dense in $\mathcal{ML}_b(\mathbb{D})$ in the Fréchet topology.*

Proof. Fix $\lambda \in \mathcal{ML}_b(\mathbb{D})$. Let λ^0 and λ^1 be the discrete and continuous parts of λ , respectively. By definition, λ^0 is the sum of Dirac measures (atoms). We identify Dirac measures appearing as terms of λ^0 with their supports (each of them is a positive number assigned to a point in \mathcal{G}).

We now fix n and partition \mathcal{G} into a locally finite, countable family of boxes $\{B'_s\}_{s=1}^\infty$ with mutually disjoint interiors such that $L(B'_s) \leq \log 2$. We enumerate the terms of λ^0 :

$$\lambda^0 = \sum_{s=1}^\infty \sum_m \mu_m^s$$

such that $\text{supp}(\mu_m^s) \subset B'_s$. If an atom belongs to the boundary side of a box, then it is shared by at least two boxes and at most four boxes. We fix one of the possible boxes to which the atom belongs and write it in the above sum only once. It is possible that $\{\mu_m^s\}_m$ consists of infinitely many Dirac measures, for any s . For each s , we take $m_{s,n}$ such that

$$(7.1) \quad \sum_{s=1}^\infty \sum_{m \geq m_{s,n}} \mu_m^k(B'_s) < 1/n.$$

Notice from the definition that

$$\lambda_n^0 := \sum_{s=1}^\infty \sum_{m \leq m_{s,n}} \mu_m^s$$

is a discrete sub-measured lamination of λ . We define a measured lamination λ_n^1 by

$$\lambda_n^1 := \lambda - \lambda_n^0 = \lambda^1 + \sum_{s=1}^\infty \sum_{m > m_{s,n}} \mu_m^k$$

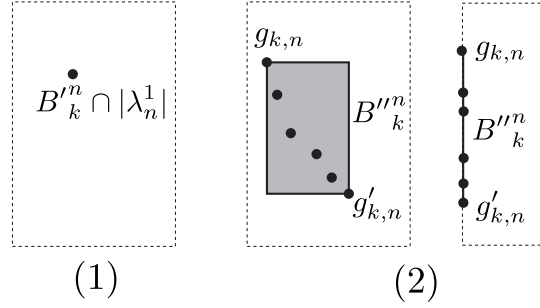
We claim the following.

Claim 1. For any n , there is a locally finite collection $\{B_k^n\}_{k=1}^\infty$ of countably many, mutually disjoint generalized boxes with the following properties.

- (1) $\{B_k^n\}_{k=1}^\infty$ covers $|\lambda_n^1|$.
- (2) $\lambda_n^1(B_k^n) < 1/n$ and $L(B_k^n) \leq \log 2$ for all k , and
- (3) extreme geodesics of B_k^n are leaves of $|\lambda_n^1|$.

Proof of Claim 1. By the definition of λ_n^1 , we can divide each B'_s into a finite collection of non-degenerate closed boxes such that its λ_n^1 -measure is less than $1/n$ and interiors of distinct boxes are disjoint. We define a sub-collection $\{B_k'^n\}_{k=1}^\infty$ to consist of all the above boxes (running all s) which intersect the support $|\lambda_n^1|$ of λ_n^1 .

We now fix one box $B_k'^n$ and modify it appropriately to get the collection of generalized boxes as in the claim.

FIGURE 4. Boxes bounded by broken lines represents B_k^n .

Case 1.1 : $B_k^n \cap |\lambda_n^1|$ **consists of one point.** When $B_k^n \cap |\lambda_n^1|$ is not an atom, then it has to belong to a boundary side B_k^n . We drop B_k^n from the family of boxes. Suppose $B_k^n \cap |\lambda_n^1|$ is an atom $\lambda'_{k,n}$ of λ , we again drop B_k^n from the collection of boxes and add $\lambda'_{k,n}$ to λ_n^0 . Since $\{B_k^n\}_{k=1}^\infty$ is locally finite, even if we continue this procedure infinitely (but countably) many times, λ_n^0 is still a locally finite sublamination of λ (cf. (1) in Figure 4).

Case 1.2 : $B_k^n \cap |\lambda_n^1|$ **contains at least two points.** Let $g_{k,n}$ and $g'_{k,n}$ be peak geodesics of $|\lambda_n^1|$ with respect to B_k^n . We replace the box B_k^n by a box $B_k''^n \subset B_k^n$ whose extreme geodesics are $g_{k,n}$ and $g'_{k,n}$ (cf. (2) in Figure 4). If it happens that $g_{k,n}$ and $g'_{k,n}$ share the same endpoint, then $B_k''^n$ is a generalized box in our sense (cf. the right figure of (2) in Figure 4).

From the definition, the family $\{B_k''^n\}_{k=1}^\infty$ of the resulting boxes is locally finite and satisfies the properties (1), (2) and (3) in the claim.

It is possible that some of the obtained closed boxes intersect along their boundaries. In this case, we divide the closed box into an open box which is the interior and into boundary sides which are generalized boxes. Each of the boundary sides is divided further into finitely many generalized boxes such that the new family of generalized boxes is pairwise mutually disjoint. Thus, after renumbering with respect to k if necessary, we finally obtain the family of generalized boxes $\{B_k^n\}_{k=1}^\infty$ as we claimed. \square

Let us continue the proof of the density theorem. Fix $n \in \mathbb{N}$. Let $\{B_k^n\}_{k=1}^\infty$ be the family of boxes from Claim 1. We fix $g_k^n \in B_k^n \cap |\lambda|$ arbitrary, and define

$$\lambda_n^2 := \sum_{k=1}^{\infty} \lambda_n^1(B_k^n) \cdot \delta_{g_k^n} \quad \text{and} \\ \lambda_n := \lambda_n^0 + \lambda_n^2,$$

where $\delta_{g_k^n}$ is the dirac measure on \mathcal{G} with support g_k^n . Since $\{B_k^n\}_{k=1}^\infty$ is locally finite, so is λ_n . Furthermore, λ_n is a measured geodesic lamination, because leaves of λ_n are leaves of λ .

We will prove that λ_n converges to λ in the Fréchet topology, which implies that discrete bounded measured laminations are dense in $\mathcal{ML}_b(\mathbb{D})$. We need the following claim to show the convergence.

Claim 2. The following holds.

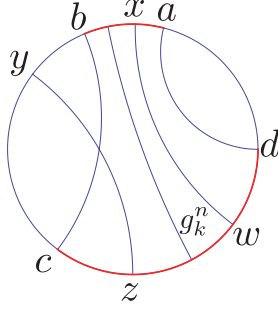


FIGURE 5. (1) in Claim 2 of the proof of Theorem 5.

- (1) For any box Q in \mathcal{G} , there are at most two boxes from the family $\{B_k^n\}_{k=1}^\infty$ such that $B_k^n \cap Q \neq \emptyset$ but $B_k^n \not\subset Q$.
- (2) The sequence $\{\lambda_n\}_{n=1}^\infty$ has uniformly bounded Thurston norms. In particular, $\lambda_n \in \mathcal{ML}_b(\mathbb{D})$.

Proof of Claim 2. (1) Let B_k^n be a box satisfying $g_k^n \in Q$ but $B_k^n \not\subset Q$. Let $Q = [a, b] \times [c, d]$ and $B_k^n = [x, y] \times [z, w]$. Without loss of generality, we may assume that b is in the interior of $[x, y]$. Then, there is no box $B_{k'}^n = I' \times J'$ such that $B_{k'}^n \cap Q \neq \emptyset$ and $I' \cap [c, z] \neq \emptyset$ or $J' \cap [c, z] \neq \emptyset$. This follows because the extreme geodesics of $B_{k'}^n$ are contained in a component of $\mathbb{D} \setminus [y, z]$ whose closure contains c , and hence, no geodesic in $B_{k'}^n$ can connect $[a, b]$ and $[c, d]$. (Figure 5). If there is another box $B_{k_1}^n = [x_1, y_1] \times [z_1, w_1]$ such that $g_{k_1}^n \in Q$ and $B_{k_1}^n \not\subset Q$, then either $a \in [x_1, y_1]$ or $d \in [x_1, y_1]$ or $a \in [z_1, w_1]$ or $d \in [z_1, w_1]$. The above reasoning implies that there could be no more boxes with the above property. Thus, there are at most two boxes with the property that $B_k^n \cap Q \neq \emptyset$ but $B_k^n \not\subset Q$.

- (2) Fix ν with $0 < \nu \leq 1$. Let $(\varphi, Q) \in \text{test}(\nu)$. From (1) of the claim, we get

$$\begin{aligned} \int_Q \varphi d\lambda_n &\leq \lambda_n^0(Q) + \sum_{g_k^n \in Q} \varphi(g_k^n) \lambda_n^1(B_k^n) \leq \lambda(Q) + \sum_{B_k^n \cap Q \neq \emptyset} \lambda_n^1(B_k^n) \\ &\leq \lambda(Q) + (\lambda(Q) + (1/n) \times 2) \leq 2 \sup_Q \lambda(Q) + 2, \end{aligned}$$

because $\varphi(g_k^n) \leq \|\varphi\|_\nu \leq 1$ and λ_n^0 is a sub-measured lamination of λ , where Q in the last term runs over all boxes with $L(Q) = \log 2$. By Lemma 2.2, we deduce that the sequence $\{\lambda_n\}_{n=1}^\infty$ has uniformly bounded Thurston norms. \square

Let us continue with the proof that λ_n converges to λ in the Fréchet topology. Let Q be a square with $L(Q) = \log 2$ and f be a continuous function on \mathcal{G} whose support is in Q^* . Let $\epsilon > 0$. We take $\delta > 0$ such that $|f(\ell) - f(\ell')| < \epsilon$ when $d(\ell, \ell') \leq \delta$, where d is the fixed metric on \mathcal{G} induced by the angle metric on S^1 with respect to $0 \in \mathbb{D}$ (cf. §2.1).

Take B_k^n with $Q \cap B_k^n \neq \emptyset$. Let $\gamma_Q^{-1}(B_k^n) = I \times J$. Suppose that $I \cap [-i, 1]$ and $J \cap [i, 1]$ are non-empty. We set

$$\hat{\lambda}_{Q,n} := (\gamma_Q)^*(\lambda_n) - d(\gamma_Q)^*(\lambda) = (\gamma_Q)^*(\lambda_n^2) - (\gamma_Q)^*(\lambda_n^1)$$

for the simplicity. We consider the following three cases for B_k^n .

Case 1. $B_k^n \subset Q$ and the length of I and J are less than δ .

In this case, we have

$$\begin{aligned} \left| \int_{\gamma_Q^{-1}(B_k^n)} f d\hat{\lambda}_{Q,n} \right| &= \left| f(\gamma_Q^{-1}(g_k^n)) \lambda_n^1(B_k^n) - \int_{\gamma_Q^{-1}(B_k^n)} f d((\gamma_Q^{-1})^*(\lambda_n^1)) \right| \\ &\leq \epsilon \lambda_n^1(B_k^n). \end{aligned}$$

Therefore, the summation over all boxes B_k^n in this case gives

$$(7.2) \quad \sum_{\{B_k^n \text{'s in Case 1}\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f d\hat{\lambda}_n \right| \leq \epsilon \lambda_n^1(Q) \leq \epsilon \lambda(Q).$$

Case 2. $B_k^n \subset Q$ and, if $\gamma_Q^{-1}(B_k^n) = I \times J$ then either I or J has length at least δ .

Notice that since Q^* is a fixed box, the number of such B_k^n in this case is $O(1/\delta)$. Hence, we have

$$(7.3) \quad \begin{aligned} \sum_{\{B_k^n \text{'s in Case 2}\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f d\hat{\lambda}_{Q,n} \right| &= O \left(\sum (\lambda_n^2(B_k^n) + \lambda_n^1(B_k^n)) \|f\|_\infty \right) \\ &= O(\|f\|_\infty / (n\delta)) \end{aligned}$$

Case 3. $B_k^n \not\subset Q$.

Notice that

$$\left| \int_{\gamma_Q^{-1}(B_k^n)} f d\hat{\lambda}_n \right| \leq (\lambda_n^2(B_k^n) + \lambda_n^1(B_k^n)) \|f\|_\infty \leq 2\|f\|_\infty / n.$$

By (1) of Claim 2, there are at most two such boxes. Hence, we have

$$(7.4) \quad \sum_{\{B_k^n \text{'s in Case 3}\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f d\hat{\lambda}_{Q,n} \right| \leq 4\|f\|_\infty / n.$$

We can now complete the proof of the convergence. Indeed, we take n sufficiently large such that $n\delta > 1/\epsilon$. Then, from the three cases above and Lemma 2.2, we conclude

$$\begin{aligned} \sup_Q \left| \int_{Q^*} f d\hat{\lambda}_{Q,n} \right| &\leq \sup_Q \left\{ \sum_{\{B_k^n \cap Q \neq \emptyset\}} \left| \int_{\gamma_Q^{-1}(B_k^n)} f d\hat{\lambda}_{Q,n} \right| \right\} \\ &\leq \sup_Q \{ \epsilon \lambda(Q) + O(\|f\|_\infty / (n\delta)) + 4\|f\|_\infty / n \} \\ &= \epsilon \left(\sup_Q \lambda(Q) \right) + O(\epsilon) = O(\epsilon), \end{aligned}$$

where the supremum is taken over all Q with $L(Q) = \log 2$. Since $\hat{\lambda}_{Q,n} = (\gamma_Q)^*(\lambda_n) - (\gamma_Q)^*(\lambda)$ and $\{\lambda_n\}_{n=1}^\infty$ is uniformly bounded, from Lemma 6.1, we have that λ_n converges to λ in the Fréchet topology. \square

Theorem 5 and Theorem 1 immediately imply Theorem 2.

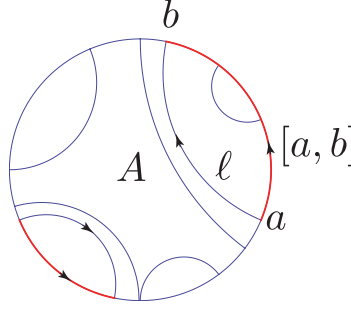


FIGURE 6. Orientations of leaves and associated intervals.

8. INFINITESIMAL EARTHQUAKES AND VECTOR FIELDS

In this section, we consider the vector fields on $\partial\mathbb{D}$ which arise by differentiating the paths of earthquakes. The aim is to prove the equivalence between the Fréchet topology on earthquake measures and the Zygmund topology on the vector fields (cf. Theorem 3) which is an analogy to Theorem 1.

8.1. Vector fields. Let λ be a bounded measured lamination. From now on, we fix a stratum A of λ such that A is either a gap or a geodesic which is not an atom of λ . Every leaf ℓ of λ is oriented as a part of the boundary of the component of $\mathbb{D} \setminus \ell$ containing A . Let a be the initial point and b the terminal point of ℓ for the given orientation. Let $[a, b]$ be an oriented interval connecting endpoints of ℓ (cf. Figure 6). Then, we set

$$\dot{E}_\ell^\lambda(z) = \begin{cases} 0 & \text{for } z \text{ outside of } [a, b] \\ \frac{(z-a)(z-b)}{a-b} & \text{for } z \in [a, b]. \end{cases}$$

When ℓ is not a leaf of λ , we put $\dot{E}_\ell^\lambda(z) = 0$ for all $z \in \partial\mathbb{D}$. For any point $z \in \partial\mathbb{D}$, $\dot{E}_\ell^\lambda(z)$ is a function of $\ell \in \mathcal{G}$.

We consider the integral

$$(8.1) \quad \dot{E}^\lambda(z) := \int_{\mathcal{G}} \dot{E}_\ell^\lambda(z) d\lambda(\ell)$$

for a measured lamination λ . For a finite lamination $\lambda = \sum_{i=1}^m \lambda_i \ell_i$, by definition, it holds

$$\dot{E}^\lambda(z) = \sum_{i=1}^m \lambda_i \dot{E}_{\ell_i}^\lambda(z).$$

One can show that the integral \dot{E}^λ in (8.1) is well-defined for all $\lambda \in \mathcal{ML}_b(\mathbb{D})$ by an approximation argument (see [5]). We give a more direct proof of the convergence of the integral in Appendix (cf. §9).

8.1.1. Infinitesimal earthquakes. For $\lambda \in \mathcal{ML}_b(\mathbb{D})$ and $t > 0$, we normalize $E^{t\lambda}$ to be the identity on the stratum A which we have fixed before. Gardiner-Hu-Lakic [7] proved that the integral (8.1) gives the tangent vector fields to the paths of earthquake deformations:

$$(8.2) \quad \dot{E}^\lambda(z) = \left. \frac{d}{dt} E^{t\lambda}(z) \right|_{t=0}$$

for $z \in \partial\mathbb{D}$ (cf. [7]). Let $\mathcal{Z}(\partial\mathbb{D})$ be the Banach space of Zygmund functions on $\partial\mathbb{D}$ modulo the subspace of quadratic polynomials (cf. §8.2). Gardiner [5] also proved the *infinitesimal earthquake theorem*, which states that the map

$$(8.3) \quad \mathcal{ML}_b(\mathbb{D}) \ni \lambda \mapsto \dot{E}^\lambda \in \mathcal{Z}(\partial\mathbb{D})$$

is bijective (Theorem 5.1 of [5]).

8.1.2. *Convergence of Vector fields.* The following proposition is well-known.

Proposition 8.1. *Let $\lambda \in \mathcal{ML}_b(\mathbb{D})$ and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathcal{ML}_b(\mathbb{D})$ with uniformly bounded Thurston norms. If $\{\lambda_n\}_{n=1}^\infty$ converges to λ in the weak* topology, then \dot{E}^{λ_n} pointwise converges to \dot{E}^λ on $\partial\mathbb{D}$.*

We shall give a proof of Proposition 8.1 in Appendix (§9.3) for the completeness. After that, we will give a simple proof of the formula (8.2) using holomorphic motions and Proposition 8.1 in §9.4.

8.2. **Fréchet and Zygmund.** Let V be a continuous function on $\partial\mathbb{D}$ satisfying $V(z)/(iz) \in \mathbb{R}$ for $z \in \partial\mathbb{D}$. We say that V is in the *Zygmund class* if there is an $M > 0$ such that

$$(8.4) \quad |V(e^{i(x+t)}) + V(e^{i(x-t)}) - 2V(e^{ix})| \leq M|t|$$

for all $0 \leq x < 2\pi$ and $0 < t < \pi$. The infimum of the constant M in (8.4) is called the *Zygmund norm* of V and we denote it by $\|V\|_{Zyg}$. Recall that $\|V\|_{Zyg} = 0$ if and only if V is a quadratic polynomial. The quotient of the class of continuous functions satisfying $V(z)/(iz) \in \mathbb{R}$ for $z \in \partial\mathbb{D}$ and inequality (8.4) by the subspace consisting of the quadratic polynomials becomes a Banach space $\mathcal{Z}(\partial\mathbb{D})$ with the norm $\|\cdot\|_{Zyg}$. We call $\mathcal{Z}(\partial\mathbb{D})$ the *Zygmund space*.

We define the *cross-ratio norm* on $\mathcal{Z}(\partial\mathbb{D})$ as follows. Let $Q = [a, b] \times [c, d]$ be a box of geodesics such that 4-points a, b, c, d lie on $\partial\mathbb{D}$ in the counter-clockwise. For $V \in \mathcal{Z}(\partial\mathbb{D})$, we set

$$V[Q] = \frac{V(a) - V(c)}{a - c} + \frac{V(b) - V(d)}{b - d} - \frac{V(a) - V(d)}{a - d} - \frac{V(b) - V(c)}{b - c}.$$

Then, the *cross-ratio norm* $\|V\|_{cr}$ of V is defined by

$$\|V\|_{cr} = \sup_Q |V[Q]|$$

where Q runs all boxes with $L(Q) = \log 2$. The Zygmund norm is equivalent to the cross-ratio norm on $\mathcal{Z}(\partial\mathbb{D})$ (see [6]).

8.3. **Proof of Theorem 3.** By Gardiner's infinitesimal earthquake theorem the map (8.3) is bijective. Hence it suffices to show that the map and its inverse are both continuous.

We first check that the map (8.3) is continuous. Let $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ in the Fréchet topology. Then $\|\lambda_n\|_{Th}$ is uniformly bounded. It follows that the sequence $V_n := \dot{E}^{\lambda_n}|_{S^1}$ has uniformly bounded cross-ratio norms. Indeed, the cross-ratio norm gives the infinitesimal change in the cross-ratios under the earthquake path $t \mapsto E^{t\lambda_n}|_{\partial\mathbb{D}}$. Assume on the contrary that $\|V_n\|_{cr} \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists a sequence Q_n of boxes in \mathcal{G} with $L(Q_n) = \log 2$ such that $|V_n[Q_n]| \rightarrow \infty$ as $n \rightarrow \infty$. Let $\gamma_{Q_n} : Q^* \mapsto Q_n$ be Möbius and let $\lambda'_n := (\gamma_n)^*(\lambda_n)$. Then there exists a subsequence of λ'_n , denoted by λ'_n for simplicity, which converges in the weak* topology to a bounded measured lamination λ' . Then, by Proposition

8.1, there exist an appropriate normalization of the earthquake vector fields such that $\dot{E}^{\lambda'_n}|_{S^1} \rightarrow \dot{E}^{\lambda'}|_{S^1}$ pointwise as $n \rightarrow \infty$. Since $|V[Q_n]| = |\dot{E}^{\lambda'_n}|_{S^1}[Q^*]| \rightarrow \infty$ as $n \rightarrow \infty$, this gives a contradiction. Thus the vector fields V_n have uniformly bounded cross-ratio norms.

A family of normalized Zygmund bounded maps whose cross-ratio norms are uniformly bounded is a normal family (see [6]). If necessary, we normalize $\dot{E}^{\lambda_n}|_{S^1}$ by adding a quadratic polynomial, such that $\dot{E}^{\lambda_n}|_{S^1}$ is a normal family. Assume on the contrary that $\dot{E}^{\lambda_n}|_{S^1} \not\rightarrow \dot{E}^{\lambda}|_{S^1}$ in the cross-ratio norm topology. Then there are $C > 0$ and a sequence of quadruples Q_n in S^1 with $L(Q_n) = \log 2$ such that $|\dot{E}^{\lambda_n}[Q_n] - \dot{E}^{\lambda}[Q_n]| \geq C$. Let γ_{Q_n} be a Möbius map such that $\gamma_{Q_n} : Q^* \rightarrow Q_n$, where $Q^* = [-i, 1] \times [i, -1]$. Then $|\gamma_{Q_n}^*(\dot{E}^{\lambda_n})[Q^*] - \gamma_{Q_n}^*(\dot{E}^{\lambda})[Q^*]| \geq C$ for all n . Since $\|\gamma_{Q_n}^*(\lambda_n)\|_{Th} = \|\lambda_n\|_{Th}$ and $\|\gamma_{Q_n}^*(\lambda)\|_{Th} = \|\lambda\|_{Th}$, it follows that the Thurston norms of $\gamma_{Q_n}^*(\lambda_n)$ and $\gamma_{Q_n}^*(\lambda)$ are uniformly bounded. Therefore, we can extract convergent subsequences of $\gamma_n^*(\lambda_n)$ and $\gamma_n^*(\lambda)$ in the weak* topology, which we denote by the same letters for simplicity. The assumption on the convergence $\lambda_n \rightarrow \lambda$ in the Fréchet topology implies that the limit of $\gamma_n^*(\lambda_n)$ equals to the limit of $\gamma_n^*(\lambda)$. On the other hand, the two sequences of vector fields $\gamma_n^*(\dot{E}^{\lambda_n})$ and $\gamma_n^*(\dot{E}^{\lambda})$ converge pointwise to different limits (even different up to addition of a quadratic polynomial) because they differ on Q^* . This implies that a single measured lamination represents two different earthquake vector fields which is impossible. Thus the map $\lambda \mapsto \dot{E}^{\lambda}|_{S^1}$ is continuous.

It remains to show that the inverse map is continuous. Assume that $\dot{E}^{\lambda_n}|_{S^1} \rightarrow \dot{E}^{\lambda}|_{S^1}$ as $n \rightarrow \infty$ in the cross-ratio norm. We claim that there exists $C > 0$ such that $\|\lambda_n\|_{Th} < C$ for all n . Suppose on the contrary that $\|\lambda_n\|_{Th} \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists a sequence I_n of closed geodesic arcs whose length is $1/n$ such that the λ_n -mass of the geodesics intersecting I_n goes to infinity as $n \rightarrow \infty$. Let l_n and r_n be the leftmost and the rightmost geodesic of $|\lambda_n|$ which intersect I_n . It is possible that $l_n = r_n$. Let γ_n be a Möbius map such that the endpoints of $\gamma_n(l_n)$ are fixed points $b < d$ in \mathbb{R} and such that the endpoints of $\gamma_n(r_n)$ converge to b and d , respectively. Let $a < b$ and $b < c < d$ be such that box $Q = [a, b] \times [c, d]$ satisfies $L(Q) = \log 2$. We normalize $\dot{E}^{(\gamma_n^{-1})^*(\lambda_n)}|_{S^1} = (\gamma_n^{-1})^*(\dot{E}^{\lambda_n}|_{S^1})$ by orienting all the leaves of $|\gamma_n(\lambda_n)|$ to the left with respect to the geodesic with endpoints (b, d) .

The cross-ratio norm is invariant under the push-forward by Möbius maps. This implies that $\|\dot{E}^{(\gamma_n^{-1})^*(\lambda_n)}|_{S^1}\|_{cr} = \|\dot{E}^{\lambda_n}|_{S^1}\|_{cr}$ is bounded. Let $V_n = \dot{E}^{(\gamma_n^{-1})^*(\lambda_n)}|_{S^1}$ for short. The normalization that we imposed on V_n gives that

$$V_n[Q] = V_n(a)\left[\frac{1}{a-c} - \frac{1}{a-b}\right] + V_n(c)\left[\frac{-1}{a-c} + \frac{-1}{c-d}\right].$$

Both terms are non-negative. Moreover, $V_n(c) \geq \lambda_n(I_n) \rightarrow \infty$ as $n \rightarrow \infty$, where $\lambda_n(I_n)$ is the λ_n -mass of geodesics intersecting I_n . Thus $V_n[Q] \rightarrow \infty$ as $n \rightarrow \infty$ which is a contradiction. Thus $\|\lambda_n\|_{Th}$ is uniformly bounded.

Assume on the contrary that $\lambda_n \not\rightarrow \lambda$ as $n \rightarrow \infty$ in the Fréchet topology. Then, after possibly taking a subsequence and renaming it, there exists a sequence Q_n of quadruples on $\hat{\mathbb{R}}$ such that $L(Q_n) = \log 2$ and

$$(8.5) \quad |\dot{E}^{\lambda_n}|_{S^1}[Q_n] - \dot{E}^{\lambda}|_{S^1}[Q_n]| \geq c > 0.$$

Let γ_n be Möbius map which maps $Q = (-a, -1, 1, a)$ onto Q_n , where $a > 1$ is chosen such that $L(Q) = \log 2$. Let $\mu_n = (\gamma_n)^*(\lambda_n)$ and $\xi_n = (\gamma_n)^*(\lambda)$. Since

$\|\mu_n\|_{Th}$ and $\|\xi_n\|_{Th}$ are uniformly bounded, there exist two subsequences of μ_n and ξ_n with common indexing which converge in the weak* topology. We can assume that μ_n and ξ_n converge in the weak* topology to μ and ξ , respectively. By (8.5) we get that $|\dot{E}^\mu|_{S^1}[Q] - \dot{E}^\xi|_{S^1}[Q]| \geq c > 0$ which implies that $\mu \neq \xi$. On the other hand, since $\dot{E}^{\lambda_n}|_{S^1} \rightarrow \dot{E}^\lambda|_{S^1}$ in the Fréchet topology, it follows that if the push-forwards of $\dot{E}^{\lambda_n}|_{S^1}$ and $\dot{E}^\lambda|_{S^1}$ by a sequence of Möbius maps pointwise converge then the limits have to be equal. This is a contradiction with $\mu \neq \xi$ by the uniqueness of the earthquake measures. Thus $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ in the Fréchet topology which is what we needed.

9. APPENDIX : THE INTEGRAL \dot{E}^λ

In this section, we consider the integral presentation of the earthquake vector field. We prove (see §9.2) that the integral in (8.1) is well-defined.

9.1. Strata and restricted measures. Recall that a stratum of a (measured) geodesic lamination λ is either a leaf of λ or the closure of a component of $\mathbb{D} \setminus \lambda$. By a *generalized stratum*, we mean either a stratum of λ or a point of $\partial\mathbb{D}$.

Let λ be a measured lamination. Let A and B be two generalized strata of λ . We denote by $\lambda_{A,B}$ a measured lamination whose support consists of leaves of λ separating A and B in \mathbb{D} , and a leaf in ∂A (resp. ∂B) facing B (resp. A), if A (resp. B) is a gap. The measure is defined to be the restriction of λ on the above set of geodesics. Thus, $\lambda_{A,B}$ is a measured geodesic lamination.

Alternatively, take a geodesic I connecting A and B where $A \cap I$ and $B \cap I$ are points. When either A or B , say B , is a point of $\partial\mathbb{D}$, we set I to be a geodesic ray from a point of A terminating at B such that $A \cap I$ consists of a point. We can define I in the similar way when both A and B are points of $\partial\mathbb{D}$. Let $|\lambda|_I$ be leaves of λ intersecting I . Notice that the set $|\lambda|_I$ is independent of the choice of the geodesic I . Since I is closed, $|\lambda|_I$ is a geodesic lamination, that is, it is a closed subset of \mathcal{G} . Hence the restriction of λ to $|\lambda|_I$ defines a Borel measure on \mathcal{G} and hence it is recognized as a measured lamination $\lambda_{A,B}$ on \mathbb{D} . When we specify the geodesic I , we denote $\lambda_{A,B}$ by λ_I .

In this notation, if B is a point of $\partial\mathbb{D}$ and $B \in \partial A$, we recognize $\lambda_I = \lambda_{A,B}$ as the zero measure. This notation will appear in Proposition 9.1.

9.2. The integral is well-defined. In this section, we prove that the integral

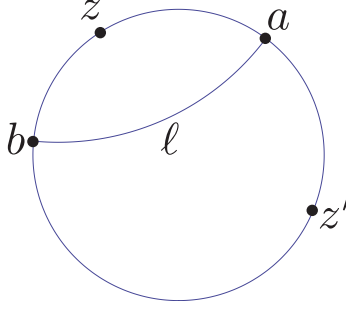
$$(9.1) \quad \int_{\mathcal{G}} \dot{E}_\ell^\lambda(z) d\lambda(\ell)$$

is well-defined for all $z \in \partial\mathbb{D}$, when $\lambda \in \mathcal{ML}_b(\mathbb{D})$.

Remark 9.1. Recall that when we fix $z \in \partial\mathbb{D}$,

$$\mathcal{G} \ni \ell \mapsto \dot{E}_\ell^\lambda(z)$$

is a function with the domain \mathcal{G} . Notice from the definition that for $z \in \partial\mathbb{D}$, $\dot{E}_\ell^\lambda(z)$ is independent of the measure λ , depends only on the support $|\lambda|$ of λ . Hence we can define $\dot{E}_\ell^\lambda(z)$ for any geodesic lamination λ .

FIGURE 7. Geodesics ℓ and ℓ' .

9.2.1. *Support of the integral.* Let A be the fixed stratum which we used to define $\dot{E}_\ell^\lambda(z)$ in §8. Let ℓ_A be the leaf of λ contained in the closure of A which is closest to z . Let z_0 be a point of ℓ_A .

Let I be the geodesic connecting z_0 and z . If $z \in \partial\mathbb{D} \cap \overline{A}$, $\dot{E}_\ell^\lambda(z)$ is identically 0 on \mathcal{G} . Hence the integral (9.1) converges in this case. Hence we may assume that z is not in \overline{A} . This means that $I \cap A = \{z_0\}$ and I is not contained in any leaf of λ .

We define a measured lamination λ_I as before. As above, we denote by $|\lambda|_I$ the support of λ_I . Namely, $|\lambda|_I = |\lambda_I| = |\lambda_{A,z}|$.

The following lemma is immediate from the definition of $\dot{E}_\ell^\lambda(z)$.

Lemma 9.1. *Suppose λ is a geodesic lamination. Then, for $z \in \partial\mathbb{D}$, the support of a function $\mathcal{G} \ni \ell \mapsto \dot{E}_\ell^\lambda(z)$ is equal to $|\lambda|_I = |\lambda_{A,z}|$.*

9.2.2. *A function \tilde{e}_z on \mathcal{G} .* For $z \in \partial\mathbb{D}$, we define a function \tilde{e}_z on \mathcal{G} as follows. Let $\ell = [a, b]$. We set

$$(9.2) \quad \tilde{e}_z(\ell) := \begin{cases} \frac{(z-a)(z-b)}{a-b} & a \neq z \text{ and } b \neq z \\ 0 & \text{otherwise,} \end{cases}$$

where in the first row of the right-hand side of (9.2), a and b are chosen such that the ordered triple (a, z, b) lies on $\partial\mathbb{D}$ counterclockwise. For instance, in Figure 7, we have $\tilde{e}_z(\ell) = \frac{(z-a)(z-b)}{a-b}$ and $\tilde{e}_{z'}(\ell) = \frac{(z'-b)(z'-a)}{b-a}$. Notice that \tilde{e}_z is well-defined and continuous on \mathcal{G} . Since $\tilde{e}_z(\ell) = \dot{E}_\ell^\lambda(z)$ on the support $|\lambda|_I$ of λ_I , by Lemma 9.1, we conclude the following.

Lemma 9.2. *Let λ be a measured lamination. Then, the function $\mathcal{G} \ni \ell \mapsto \dot{E}_\ell^\lambda(z)$ is measurable with respect to λ . Furthermore, for any $z \in \partial\mathbb{D}$, if the geodesic ray I above is not contained in any leaf of λ , it holds*

$$(9.3) \quad \int_{\mathcal{G}} \dot{E}_\ell^\lambda(z) d\lambda(\ell) = \int_{\mathcal{G}} \tilde{e}_z(\ell) d\lambda_I(\ell) = \int_{\mathcal{G}} \tilde{e}_z(\ell) d\lambda_{A,z}(\ell),$$

if either the middle term or the right-hand side of (9.3) are defined.

In particular, the integral (9.1) is represented as the integration of a continuous function defined independently of λ , but depending only on z . Thus, to check the convergence of the integral (9.1), we may prove the integrability of \tilde{e}_z with respect to $\lambda_{A,z}$.

We now give properties of the function \tilde{e}_z . One can easily see that

$$\tilde{e}_{T(z)}(T(\ell))T'(z)^{-1} = \tilde{e}_z(\ell)$$

for all $\ell \in \mathcal{G}$, $z \in \partial\mathbb{D}$ and $T \in \text{Möb}(\mathbb{D})$. Let J be the radial geodesic ray emanating from 0 to $z \in \partial\mathbb{D}$. Let w_d ($d \geq 0$) be the length parametrization of J with $w_0 = 0$. The function \tilde{e}_z has the following property.

Lemma 9.3. *Let $z \in \partial\mathbb{D}$. For $D_0 > 0$, it holds*

$$|\tilde{e}_z(\ell)| \leq (8 \cosh(D_0))e^{-d}$$

when ℓ intersects the D_0 -neighborhood of w_d .

Proof. Notice that the set $K_0 \subset \mathcal{G}$ of all geodesics intersecting the hyperbolic disk of center 0 and radius D_0 is compact. By a hyperbolic trigonometry formula, we have

$$|\tilde{e}_z(\ell)| = |(z-a)(z-b)|/|a-b| \leq 4/|a-b| \leq 8 \cosh(D_0)$$

for all $\ell = [a, b] \in K_0$ and $z \in \partial\mathbb{D}$.

Let ℓ be a geodesic which intersects the D_0 -neighborhood of w_d . Let T be a Möbius transformation acting on \mathbb{D} with $T(w_d) = 0$ and fixing z . Since w_d is on J , $w_d = |w_d|z$. Since $T(\ell) \in K_0$, we obtain

$$\begin{aligned} |\tilde{e}_z(\ell)| &= |\tilde{e}_{T(z)}(T(\ell))| |T'(z)|^{-1} \leq (8 \cosh(D_0)) |1 - \overline{w_d}z|^2 / (1 - |w_d|^2) \\ &= (8 \cosh(D_0)) \frac{1 - |w_d|}{1 + |w_d|} = (8 \cosh(D_0))e^{-d}, \end{aligned}$$

which implies what we wanted. \square

9.2.3. Proof that the integral is well-defined. Recall that A is the stratum which we fixed in the beginning and $z_0 \in A$ is the initial point of I . Let z_d ($d \geq 0$) be the length parametrization of I . We set $I_d = \{z_k \mid k \geq d\}$. We can define a measured lamination λ_{I_d} as above. Notice that if $|\lambda|_I$ contains no leaves which diverge in \mathcal{G} , the support $|\lambda|_I$ of λ_I is compact and eventually λ_{I_d} becomes the zero measure.

The integral (9.1) for bounded measured laminations converges because of the following estimate.

Proposition 9.1 (Rate of decay). *Let $\lambda \in \mathcal{ML}_b(\mathbb{D})$ and $z \in \partial\mathbb{D}$. Let ℓ_A be the leaf of λ in A facing z . Let $z_0 \in \ell_A$ and I be the geodesic ray emanating from z_0 and terminating at z as above. Then, there is a constant C_2 depending only on the hyperbolic distance between 0 and z_0 such that*

$$(9.4) \quad \int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_d}(\ell) \leq C_2 \|\lambda\|_{Th} \cdot e^{-d}$$

for $d \geq 0$.

Proof. When z is in the closure of A , the interval I is contained in A . Hence λ_I is the zero measure, and (9.4) holds for all $d \geq 0$. In this case $\dot{E}_\ell^\lambda(z)$ is identically zero on \mathcal{G} . Therefore, the integral in (9.1) converges and equals to zero (and the equation (9.3) also holds). Hence we may assume that $z \in \partial\mathbb{D} \setminus \overline{A}$. This assumption means that I transversely intersects some leaves of λ in \mathbb{D} . However, note that z may be an endpoint of some leaf of λ .

Let $\{I_{n,d}\}_{n=0}^\infty$ be a sequence of consecutive subintervals of I_d such that $z_d \in I_{0,d}$ and $I_{n,d} \cap I_{n+1,d} = \{z_{d+n}\}$. Notice that each $I_{n,d}$ has unit length. We define a measured sublamination $\lambda_{I_{n,d}}$ of λ_I as above. When there is no leaf of λ intersecting $I_{n,d}$, we define $\lambda_{I_{n,d}}$ to be the zero measure as we noted before.

As in Lemma 9.3, we denote by J the radial geodesic ray emanating from 0 to z , and w_d ($d \geq 0$) the length parametrization of J with $w_0 = 0$. Let ℓ be a leaf of $\lambda_{I_{n,d}}$

and $\{z_{d'}\} = \ell \cap I_{n,d}$. Then, by the triangle inequality, we have $d_{\mathbb{D}}(0, z_{d'}) \geq n + d - D_0$. Since J shares the endpoint z with I , $d_{\mathbb{D}}(w_{d'}, z_{d'}) \leq d_{\mathbb{D}}(z_0, w_0) = D_0$, which means that any leaf of $\lambda_{I_{n,d}}$ intersects the $D_0 + 1$ -neighborhood of $w_{d'}$. By Lemma 9.3, we have

$$|\tilde{e}_z(\ell)| \leq (8 \cosh(D_0 + 1))e^{-d_{\mathbb{D}}(0, z_{d'})} \leq (8 \cosh(D_0 + 1))e^{-(d+n-D_0)} = C_1 e^{-(d+n)},$$

where $C_1 = 8e^{D_0} \cosh(D_0 + 1)$. Therefore, we get

$$\begin{aligned} \int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_{n,d}}(\ell) &\leq C_1 e^{-(d+n)} \lambda_{I_{n,d}}(\mathcal{G}) = C_1 e^{-(d+n)} \lambda_{I_{n,d}}(I_{n,d}) \\ &\leq C_1 \|\lambda\|_{Th} e^{-d} \cdot e^{-n}, \end{aligned}$$

since each $I_{n,d}$ has unit length and the support of $\lambda_{I_{n,d}}$ is contained in $I_{n,d}$. Thus, we conclude

$$\int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_d}(\ell) \leq \sum_{n=0}^{\infty} \int_{\mathcal{G}} |\tilde{e}_z(\ell)| d\lambda_{I_{n,d}}(\ell) \leq C_2 \|\lambda\|_{Th} e^{-d},$$

where $C_2 = (1 - e^{-1})C_1$. \square

9.3. Weak* convergence and Pointwise convergence. In this section, we prove the continuity of the integral (8.1) on $\mathcal{ML}_b(\mathbb{D})$ with respect to the weak* topology.

Proposition 9.2 (Pointwise convergence). *Fix α with $0 \leq \alpha < 1$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of measured laminations which converges in the weak* topology to a measured lamination $\lambda \in \mathcal{ML}_b(\mathbb{D})$. If the Thurston norms of the sequence $\{\lambda_n\}_{n=1}^{\infty}$ of measured laminations are uniformly bounded, then there is a choice of normalizations for \dot{E}_{ℓ}^{λ} and $\dot{E}_{\ell}^{\lambda_n}$ such that*

$$\lim_{n \rightarrow \infty} \int_{\mathcal{G}} \dot{E}_{\ell}^{\lambda_n}(z) d\lambda_n(\ell) = \int_{\mathcal{G}} \dot{E}_{\ell}^{\lambda}(z) d\lambda(\ell)$$

for all $z \in \partial\mathbb{D} = S^1$.

Proof. The proof follows the same outline as the proof of [14, Lemma 3.2]. We first fix the normalizations of \dot{E}_{ℓ}^{λ} and $\dot{E}_{\ell}^{\lambda_n}$. Let A be a fixed stratum of λ which is either a gap of λ or a leaf of λ whose λ -measure is zero (i.e. A is not an atom of λ). Let $z_0 \in A$ be a point in the interior of A if it is a gap, or any point of A if it is a leaf of λ . Let A_n be the stratum of λ_n which contains z_0 . We orient each $\ell \in |\lambda|$ to the left as seen from A . If A is a geodesic, then we orient A arbitrary. This gives a well-defined function \dot{E}_{ℓ}^{λ} for $\ell \in |\lambda|$ which in turn implies

$$\int_{\mathcal{G}} \dot{E}_{\ell}^{\lambda}(z) d\lambda(\ell) = \int_{\mathcal{G}} \tilde{e}(\ell) d\lambda_{A,z}(\ell).$$

We define $\dot{E}_{\ell}^{\lambda_n}$ by giving the left orientation to each ℓ with respect to the stratum A_n in the same fashion.

Let I be a geodesic ray from z_0 to z and let $z_d \in I$ be such that the distance between z_0 and z_d is $d \geq 0$. We fix $d > 0$ such that z_d is contained in a stratum A_d of $|\lambda|$ which is either a gap or a leaf which is not an atom of λ .

Given $i \in \mathbb{N}$, let $I_i = (z_l^i, z_r^i)$ be an open geodesic arc whose endpoints are on the distance $1/i$ from z_0 and z_d , and which contains z_0, z_d . The set of geodesics of \mathbb{D} which intersect I_i is open in \mathcal{G} and contains all geodesics of $|\lambda|$ which intersect the closed geodesic arc with endpoints z_0 and z_d . Since the lengths of (z_l^i, z_0)

and (z_d, z_r^i) are going to zero as $i \rightarrow \infty$, it follows that the λ -measure of the set of geodesics intersecting (z_l^i, z_0) and (z_d, z_r^i) is going to zero as $i \rightarrow \infty$ by the choice of z_0 and z_d (namely, A and A_{z_d} are either gaps or non-atomic leaves). Let $\varphi_i : \mathcal{G} \rightarrow \mathbb{R}$ be a non-negative continuous function whose support consists of geodesics intersecting $I_i = (z_l^i, z_r^i)$ and which is identically equal to 1 on the set of geodesics intersecting $[z_0, z_d]$. Then the function $\ell \mapsto \varphi_i(\ell)\tilde{e}_\ell(z)$ is a continuous function on \mathcal{G} with compact support. It follows that

$$\int_{\mathcal{G}} \varphi_i(\ell)\tilde{e}_\ell(z)d\lambda_n(\ell) \rightarrow \int_{\mathcal{G}} \varphi_i(\ell)\tilde{e}_\ell(z)d\lambda(\ell)$$

as $n \rightarrow \infty$ by the weak* convergence $\lambda_n \rightarrow \lambda$.

Note that

$$\int_{\mathcal{G}} \varphi_i(\ell)\tilde{e}_\ell(z)d\lambda_n(\ell) \leq \int_{\mathcal{G}} |\tilde{e}_\ell(z)|d[(\lambda_n)_{(z_l^i, z_0)} + (\lambda_n)_{(z_d, z_r^i)}](\ell) + \int_{\mathcal{G}} \tilde{e}_\ell(z)d(\lambda_n)_{(z_0, z_d)}(\ell)$$

and

$$\int_{\mathcal{G}} \varphi_i(\ell)\tilde{e}_\ell(z)d\lambda(\ell) \leq \int_{\mathcal{G}} |\tilde{e}_\ell(z)|d[\lambda_{(z_l^i, z_0)} + \lambda_{(z_d, z_r^i)}](\ell) + \int_{\mathcal{G}} \tilde{e}_\ell(z)d\lambda_{(z_0, z_d)}(\ell).$$

The choice of z_0 and z_d is such that the total masses of $\lambda_{(z_l^i, z_0)}$ and $\lambda_{(z_d, z_r^i)}$ on \mathcal{G} converge to zero as $i \rightarrow \infty$. Since λ_n converges to λ in the weak* sense, it follows that given $\epsilon > 0$ there exist $i_0, n_0 \in \mathbb{N}$ such that the total masses of $\lambda_{(z_l^i, z_0)}$, $\lambda_{(z_d, z_r^i)}$, $(\lambda_n)_{(z_l^i, z_0)}$ and $(\lambda_n)_{(z_d, z_r^i)}$ on \mathcal{G} are less than ϵ for $i \geq i_0$ and $n \geq n_0$. The above three inequalities imply that

$$\int_{\mathcal{G}} \tilde{e}_\ell(z)d(\lambda_n)_{(z_0, z_d)}(\ell) \rightarrow \int_{\mathcal{G}} \tilde{e}_\ell(z)d\lambda_{(z_0, z_d)}(\ell)$$

as $n \rightarrow \infty$.

Since $|\int_{\mathcal{G}} \tilde{e}_\ell(z)d(\lambda_n)_{(z_0, z_d)}(\ell) - \int_{\mathcal{G}} \tilde{e}_\ell(z)d\lambda_{(z_0, z_d)}(\ell)| \leq Ce^{-d}$ and $|\int_{\mathcal{G}} \tilde{e}_\ell(z)d\lambda_{(z_0, z_d)}(\ell) - \int_{\mathcal{G}} \tilde{e}_\ell(z)d\lambda(\ell)| \leq Ce^{-d}$, the conclusion follows. \square

9.4. Differentiation of earthquake paths. In this section, we reprove the formula (8.2).

9.4.1. Holomorphic motions and Complex earthquakes. Let S be a subset of $\hat{\mathbb{C}}$ and let D be a domain in $\hat{\mathbb{C}}$. A *holomorphic motion of S over D with base point $t_0 \in D$* is, by definition, a map $h : S \times D \rightarrow \hat{\mathbb{C}}$ satisfying the following three properties:

- (1) $h(x, t_0) = x$ for all $x \in S$.
- (2) For all $t \in D$, $h_t(\cdot) := h(\cdot, t)$ is injective on S .
- (3) For all $s \in S$, $h(s, \cdot) : D \rightarrow \hat{\mathbb{C}}$ is holomorphic.

By Slodkowski's theorem ([19]), if D is conformally equivalent to the unit disk, any holomorphic motion h of S over D with base point $t_0 \in D$ extends to a holomorphic motion \tilde{h} of $\hat{\mathbb{C}}$ over D and for each $t \in D$, \tilde{h}_t is K_t -quasiconformal mapping where $K_t = \exp(d_D(t_0, t))$ and d_D is the Poincaré distance on D normalized such that it has curvature -1 .

The following theorem is proved in [12].

Theorem 6 (Theorem 2 in [12]). *Let $\lambda \in ML_b(\mathbb{D})$. The earthquake map $(z, t) \mapsto E^{t\lambda}(z)$ for $t > 0$ and $z \in \partial\mathbb{D}$ extends to a holomorphic motion $(z, \tau) \mapsto E^{\tau\lambda}(z)$ of $\partial\mathbb{D}$ over a neighborhood S_λ of \mathbb{R} in \mathbb{C} with base point $\tau = 0$.*

The domain S_λ in the theorem above is concretely defined by

$$(9.5) \quad S_\lambda = \{\tau = t + is \mid |s| < \epsilon_0 / [C_0 \exp(\|t\lambda\|_{Th}) \|\lambda\|_{Th}]\},$$

where ϵ_0 and C_0 are independent of λ .

Proof of Proposition 8.1. We first show the convergence in the case when $\{\lambda_n\}_{n=1}^\infty$ is a finite approximation of λ . From the proof of Theorem 2 in [12], we know that there is a neighborhood V_0 of $\partial\mathbb{D}$ such that the complement of V_0 contains at least 3 points and $E^{\tau\lambda_n}(z) \in V_0$ for all $\tau \in S_\lambda$, $z \in \partial\mathbb{D}$ and $n \in \mathbb{N}$, where we assume in the definition that the restriction of $E^{t\lambda_n}$ is the identity on a stratum of λ_n containing A . This implies that $\{E^{\tau\lambda_n}(z)\}_{\tau \in S_\lambda}$ is normal family and converges to $E^{\tau\lambda}(z)$ on any compact set of S_λ . From the Weierstrass' theorem, we have

$$\left. \frac{d}{d\tau} E^{\tau\lambda}(z) \right|_{\tau=0} = \lim_{n \rightarrow 0} \left. \frac{d}{d\tau} E^{\tau\lambda_n}(z) \right|_{\tau=0}.$$

On the other hand, by Theorem 9.2, the integral in (8.1) varies continuously on $\mathcal{ML}_b(\mathbb{D})$. Hence, we get the formula (8.2). \square

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